Field Theory Approach to the Dynamics of Gapped Quantum Spin Chains

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Outline

1. Why low-D Quantum Spin Systems are interesting.

2. Some Gapped Quantum Spin Chains and their Quantum Field Theory Limits.

3. Integrable QFTs.

4. T=0 Dynamical Response Functions in integrable QFTs.

5. T>0 Dynamical Response Functions in integrable QFTs.

6. Nonequilibrium Dynamics.

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Quantum Spin Systems are inherently **strong coupling** problems:

correlated electron models → Quantum Spin Models

strong interaction limit

$$H = -t \sum_{\langle j,k \rangle,\sigma} c^{\dagger}_{j,\sigma} c_{k,\sigma} + \text{h.c.} + U \sum_{j} n_{j,\uparrow} n_{j,\downarrow}$$

→ large U/t →

$$H' = \frac{4t^2}{U} \sum_{\langle j,k \rangle} \mathbf{S}_j \cdot \mathbf{S}_k$$

Why (Low-D) Quantum Spin Systems are interesting:

Quantum Spin Systems are inherently **strong coupling** problems:

strong interactions are interesting as they are expected to lead to new **collective** behaviour

Dimensionality

D=3

"Conventional Behaviour": **Spontaneous Symmetry Breaking** of spin rotational symmetry at low T; Physics of Long-Range Order (spinwaves)

Spinwaves determine physics over large range of T and E (even far above T_N)

Basic physics is well understood.

→ Look at **Frustrated Systems** to avoid simple SSB scenario; but physics usually well described classically.

Dimensionality

D=2

(layered materials)

Quantum Fluctuations are **stronger**: SSB only at T=0 (Mermin-Wagner).

Frustrated systems may realize new **quantum** states of matter: **Quantum Spin Liquids**

⟶ **very interesting**, but difficult to address theoretically by analytic or numeric means.

Dimensionality

D=1

(chain materials)

Quantum Fluctuations are **very strong**: SSB not even at T=0 (Mermin-Wagner).

Exotic Quantum Spin Liquid States are abundant in D=1.

- **Special techniques** (integrability, bosonization, DMRG, insert your favourite method here) allow **in depth analysis**.
- High-resolution experiments on **dynamical properties** for many materials.

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Paradigm 1: spin-1/2 Heisenberg Chain

- **Gapless spin liquid** (ground state disordered for all T≥0) Hulthen '38
- Excitations: fractionalized spin-1/2 objects "spinons" Faddeev&Takhtajan '84
- T=0: **quantum critical system** (power-law decays of correlations)
- low energy properties: **Luttinger Liquid** Luther&Peschel '75 Haldane '81

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Extremely interesting! Remains object of much recent work (e.g. threshold singularities).

Glazman, Imambekov, Pustilnik, Khodas,... Affleck, Pereira, White, Sirker... Zvonarev, Cheianov, Giamarchi '06 –'11

A different talk !

Gapped Quantum Spin Chains & Ladders

A. Integer Spin Heisenberg Chains (S=1,2,3,...)

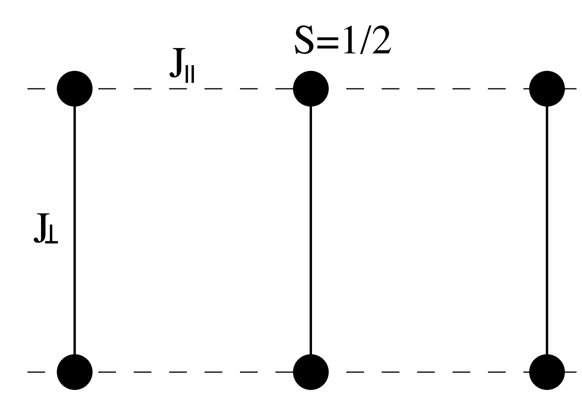
Haldane '83, Affleck '90

Kenzelmann et. al. '01
Zaliznyak et. al. '01
Zheludev et. al. '04
Xu et. al. '07

$$H = J\sum_n S_n^x S_{n+1}^x + S_n^y S_{n+1}^y + S_n^z S_{n+1}^z \,, \quad \mathbf{S}_j^2 = S(S+1).$$

$CsNiCl_3$, NDMAP, $YBaNiO_5$...

B. 2-Leg Heisenberg Spin-1/2 **Ladders**



$$H = J_{\|}\sum_{a=1}^{2}\sum_j \mathbf{S}_{a,j}\cdot\mathbf{S}_{a,j+1} + J_{\perp}\sum_j \mathbf{S}_{1,j}\cdot\mathbf{S}_{2,j}$$

CuNitrate, $(C_5H_{12}N)_2CuBr_4$, $CaCu_2O_3$, $La_4Sr_{10}Cu_{24}O_{41}$, DIMPY ...

Dagotto et. al. '92, Shelton et. al. '96, Schmidt&Uhrig '05 ...

Notbohm et. al. '07; Ruegg et. al. '08, Lake et. al. '10, Thielemann et. al. '09 Bouillot et. al. '11, Tennant et. al. '12, Schmidiger et. al. '12 ...

C. Field-Induced Gap spin-1/2 chain materials

$$\mathcal{H} = \sum_j J\mathbf{S}_j \cdot \mathbf{S}_{j+1} - HS_j^z - h(-1)^j S_j^x \ .$$

Dender et al '97, Asano et al '00, '02, Feyerherm et al '00, Kohgi et al '01, Zvyagin et al '04, '05...

Oshikawa&Affleck '97, '02
Essler&Tsvelik '97, Essler '99, ...

CuBenzoate, CDC, Cu-Pyrimidine, Yb_4As_3,...

D. Transverse Field Spin-1/2 Chain

$$\mathcal{H} = \sum_j J[S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \delta S_j^z S_{j+1}^z] - HS_j^x \ .$$

Nagler et al '82; Kenzelmann et al '02; Oosawa et al '06

Dmitriev, Krivnov&Ovchinnikov '02
Caux, Essler&Loew '03
Coldea et al (unpublished)

Cs_2CoCl_4 , $CsCoBr_3$, $CsCoCl_3$, $TlCoCl_3$, ...

E. J_1–J_2 Model a.k.a. 2-leg zigzag ladder

$$\mathcal{H} = J_1 \sum_j \mathbf{S}_j \cdot \mathbf{S}_{j+1} + J_2 \sum_j \mathbf{S}_j \cdot \mathbf{S}_{j+2}.$$

$J_2 > 0.2411 J_1 > 0$

Haldane '82, White&Affleck '96, Eggert '96, Nersesyan et al '98, ...

$SrCuO_2$, $LiCuVO_4$ (ferromagnetic J_1),...

F. Dimerized Spin-1/2 Chain

Uhrig&Schulz '96, Essler, Tsvelik&Delfino '96 Knetter&Uhrig '00, Orignac '04,...

$$\mathcal{H} = J \sum_n [1 - (-1)^n \delta]\, \mathbf{S}_n \cdot \mathbf{S}_{n+1}$$

$CuGeO_3$

G. Transverse Field Ising Chain

Coldea et al '10

Pfeuty '70, Wu et al '76, Vaidya&Tracy '78 Cardy&Mussardo '90, Yurov&Zamolodchikov '91

$$\mathcal{H} = -J \sum_j S_j^x S_{j+1}^x + h S_j^z$$

$CoNb_2O_6$

Field Theory Limit of Quantum Spin chains

Spin-S Heisenberg model with **S>>1**.

Haldane '83,
Affleck '89

staggered component: $n^a_{2i+1/2} = (S^a_{2i+1} - S^a_{2i})/2S$,

smooth component: $M^a_{2i+1/2} = (S^a_{2i+1} + S^a_{2i})/2.$

Constraint:

$$\mathbf{n}\cdot\mathbf{n} = 1 + \frac{1}{s} - \frac{\mathbf{M}\cdot\mathbf{M}}{s^2} \approx 1 \ , \qquad \mathbf{n}\cdot\mathbf{M} = 0 \ .$$

Naive Continuum Limit: $n^a_{2i+1/2} \rightarrow n^a(x),\ M^a_{2i+1/2} \rightarrow a_0 M^a(x),$

$$H = \frac{v}{2}\int dx \left[g\left(\mathbf{M} + \frac{\theta}{4\pi}\partial_x \mathbf{n}\right)^2 + \frac{1}{g}\left(\partial_x \mathbf{n}\right)^2\right] \ , \ \theta = 2\pi S \ , \ g = 2/S \ , \ v = JSa_0.$$

Commutators:

$$[M^a(x), M^b(y)] = i\epsilon^{abc}\, M^c(x)\, \delta(x-y)\ ,$$
$$[M^a(x), n^b(y)] = i\epsilon^{abc}\, n^c(x)\, \delta(x-y)\ ,$$
$$[n^a(x), n^b(y)] = \frac{1}{S^2} i\epsilon^{abc}\, M^c(x)\, \delta(x-y) \simeq 0.$$

solved by

$$M^a(x) = \frac{1}{vg}\, \mathbf{n}(x) \times \frac{\partial \mathbf{n}(x)}{dt}.$$

O(3) nonlinear sigma model

$$\mathcal{L} = \frac{v}{2g}\, \partial_\mu \mathbf{n} \cdot \partial^\mu \mathbf{n}$$

Constraint: n·n=1

$$v = JSa_0.$$
$$g = 2/S$$

When does this QFT describe the lattice model ?

- large distances, late times (for dynamics)
- low energies: $\omega \ll J$
- gap much less than cutoff → $\Delta \ll J$

C. Field-Induced Gap spin-1/2 chain materials

$$\mathcal{H} = \sum_j J\mathbf{S}_j \cdot \mathbf{S}_{j+1} - HS_j^z - h(-1)^j S_j^x .$$

$\longrightarrow$

$$\mathcal{H} = \frac{v}{16\pi}\int dx \left[(\partial_x\Phi)^2 + (\partial_x\Theta)^2\right] - \mu(h)\int dx \cos(\beta\Theta).$$

Sine-Gordon model

$$\partial_x\Theta = -\partial_t\Phi$$

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$$\beta = \beta(H)$$

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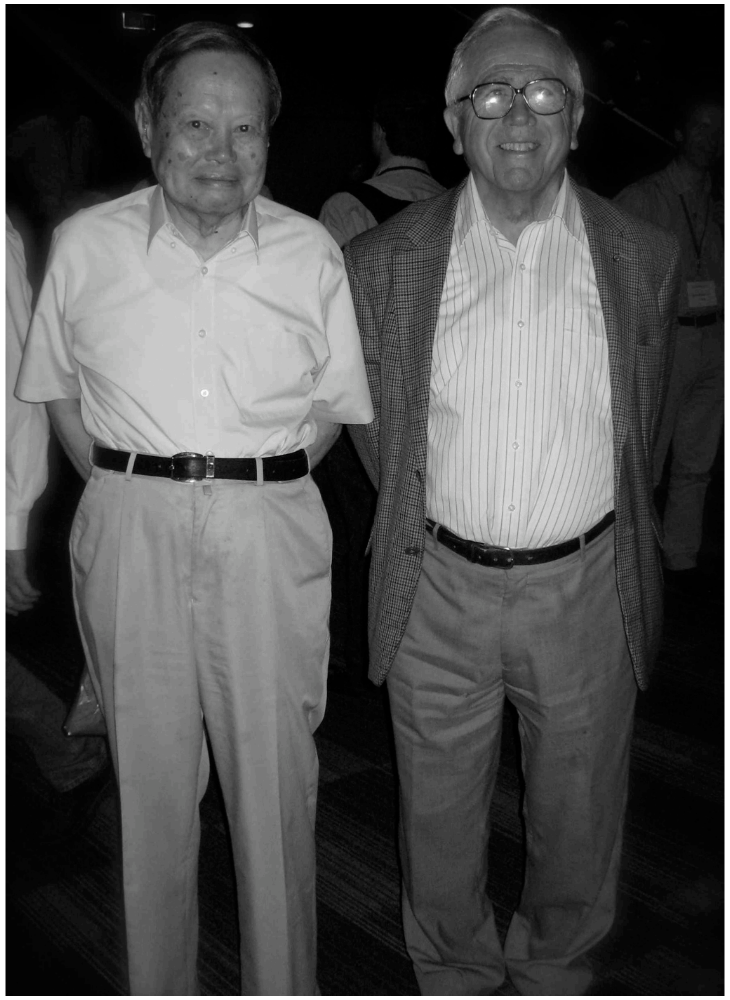
C.N. Sine →



← R. Gordon

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G. Transverse Field Ising Chain

$$\mathcal{H} = -J\sum_j S_j^x S_{j+1}^x + hS_j^z \longrightarrow H = \int_{-\infty}^{\infty}\frac{dx}{2\pi}\left[\frac{iv}{2}(\bar\psi\partial_x\bar\psi - \psi\partial_x\psi) - i\Delta\psi\bar\psi\right]$$

Majorana (real) fermion

B. 2-Leg Heisenberg Spin-1/2 **Ladders**

$$H = J_{\parallel}\sum_{a=1}^{2}\sum_j \mathbf{S}_{a,j}\cdot\mathbf{S}_{a,j+1} + J_{\perp}\sum_j \mathbf{S}_{1,j}\cdot\mathbf{S}_{2,j} \longrightarrow \mathcal{H} = \sum_{a=1}^{4}\int\frac{dx}{2\pi}\left[\frac{iv}{2}(\bar\psi_a\partial_x\bar\psi_a - \psi_a\partial_x\psi_a) - i\Delta_a\psi_a\bar\psi_a\right]$$

$$\Delta_1 = \Delta_2 = \Delta_3 = -\frac{\Delta_4}{\sqrt{3}}$$

4 "Ising models"

These relativistic QFTs are **integrable** (the lattice models are not!)

⟶ use integrability to determine
dynamical
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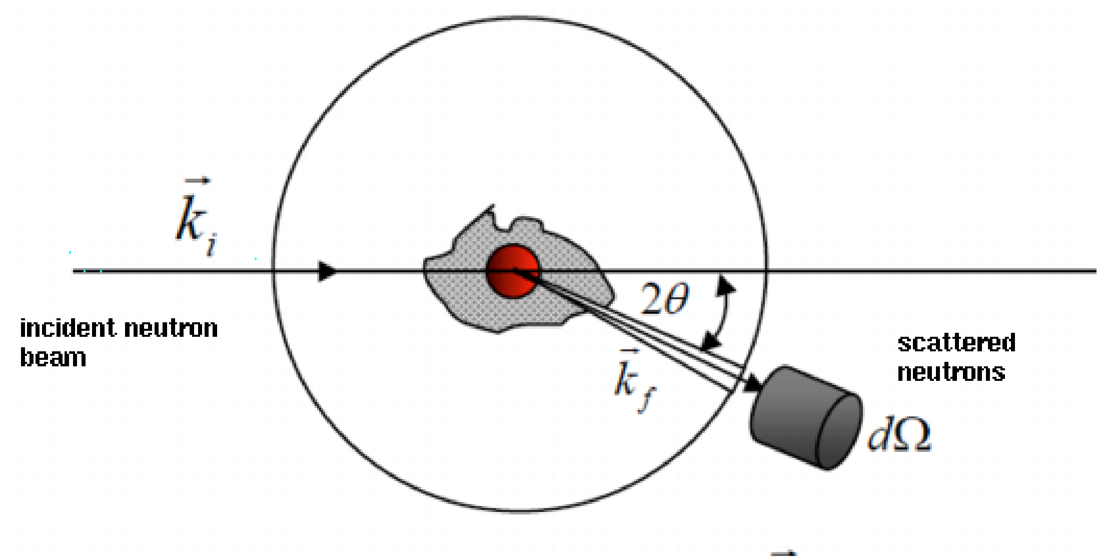
These relativistic QFTs are **integrable** (the lattice models are not!)

→ use integrability to determine dynamical correlation functions

Which ones?

Scattering Experiments

Scattering Experiments (neutrons, light, electrons) measure **imaginary parts** of **retarded** 2-point functions of local operators



Inelastic neutron scattering experiments measure

$$\frac{d^2\sigma}{d\Omega d\omega} \propto S(\underbrace{\omega, q}, T) \propto \int_0^\infty dt e^{i\omega t} \sum_j e^{-iqja_0} \frac{\mathrm{tr}\left(e^{-H/T}[S^a_{j+n}(t), S^a_n]\right)}{\mathrm{tr} e^{-H/T}}.$$

energy/mtm transferred to sample

What we want to calculate: Spin-S Heisenberg Chain

$$H = J \sum_n \mathbf{S}_n \cdot \mathbf{S}_{n+1} \ , \quad \mathbf{S}_n^2 = S(S+1)$$

large integer S, low energies

$$\mathbf{S}_j \approx S(-1)^{ja_0}\mathbf{n}(x) + \mathbf{M}(x)$$

$$\mathbf{M}(x) = \frac{1}{vg}\mathbf{n}(x) \times \frac{\partial \mathbf{n}(x)}{\partial t}$$

O(3) nonlinear sigma model

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$$g = 2/S$$

Susceptibility

around k=π/a_0:

$$\chi(\omega, k = \frac{\pi}{a_0} + q) \propto -i \int dt\ dx\ e^{i\omega t - iqx} \langle [n^a(t,x), n^a(0,0)] \rangle_T$$

around k=0:

$$\chi(\omega, q) \propto -i \int dt\ dx\ e^{i\omega t - iqx} \langle [M^a(t,x), M^a(0,0)] \rangle_T$$

$$|qa_0| \ll \pi$$

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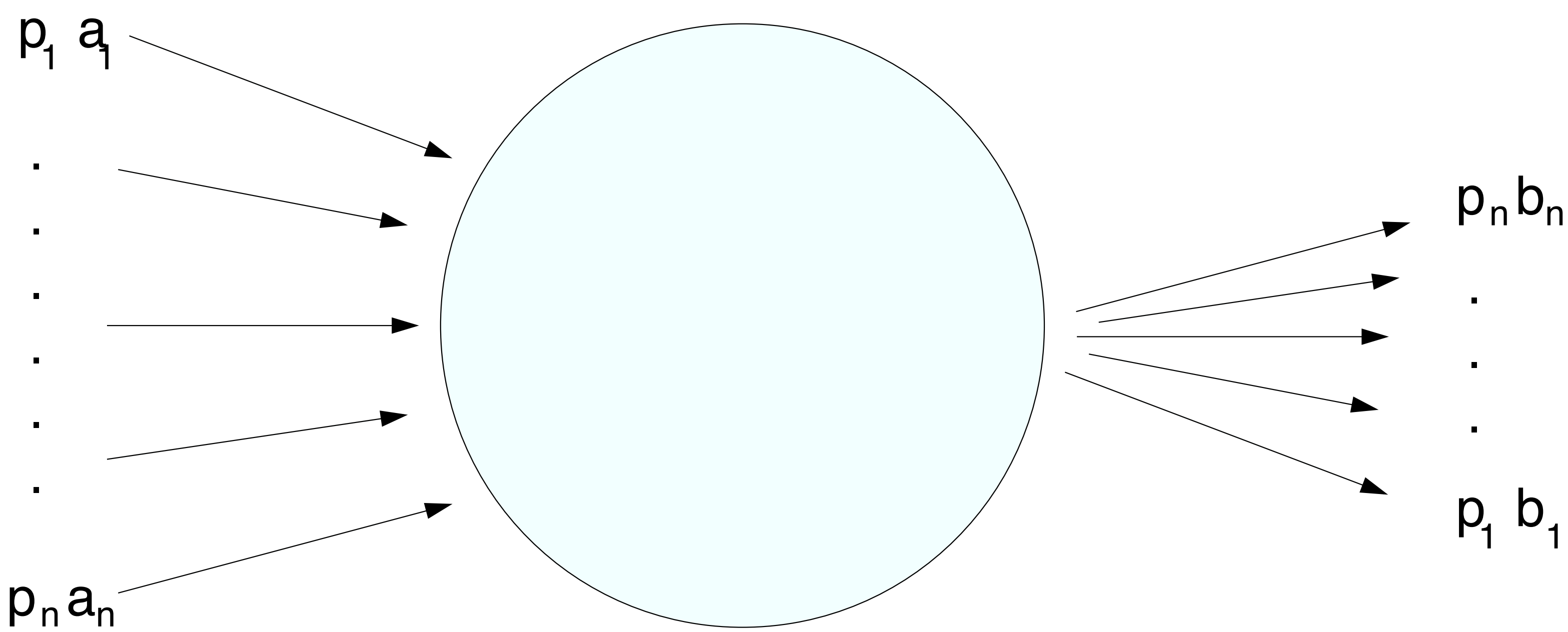
Integrable QFTs

Elementary excitations scatter **purely elastically**



all momenta conserved **individually!**

Basis States: $|n\rangle = |p_1,\ldots,p_n\rangle_{a_1\ldots a_n}$

Total Momentum: $P_n = \Sigma_{j=1}\ p_j$

Total Energy: $E_n = \Sigma_{j=1}\ \varepsilon(p_j) \geq n\Delta$

Remarks:

1. Elementary excitations usually very complicated in terms of the fields defining the theory (solitons rather than modes of field)

2. Elementary excitations are **neither bosons nor fermions**: generalized commutation relations (Faddeev-Zamolodchikov algebra)

$$Z_a^\dagger(p) Z_b^\dagger(q) = S_{ab}^{cd}(p,q) Z_c^\dagger(q) Z_d^\dagger(p)$$

interpolates between fermions (|p-q| small) and bosons (|p-q| large)

"Form Factor Bootstrap Approach"

(**Karowski**/Weisz '78, **Smirnov** '93, **Lukyanov** '95, Delfino/Mussardo '95, Balog/Niedermaier '97, Babujian/Karowski '99...)

scattering factorizable

→ consistency →

Exact 2-particle S-matrix

→ "crossing" "Watson's equations" "minimal analyticity" →

Exact Matrix elements of local operators

$\langle n|\, \mathcal{O}(0,0)\, |m\rangle$

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T=0 Dynamical Structure Factor

$$S(\omega, k = \frac{\pi}{a_0} + q) \propto \int dt\ dx\ e^{i\omega t - iqx} \langle 0|n^a(t,x) n^a(0,0)|0\rangle$$

T=0 Dynamical Structure Factor

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$\Sigma_m\ |m\rangle\langle m|$

T=0 Dynamical Structure Factor

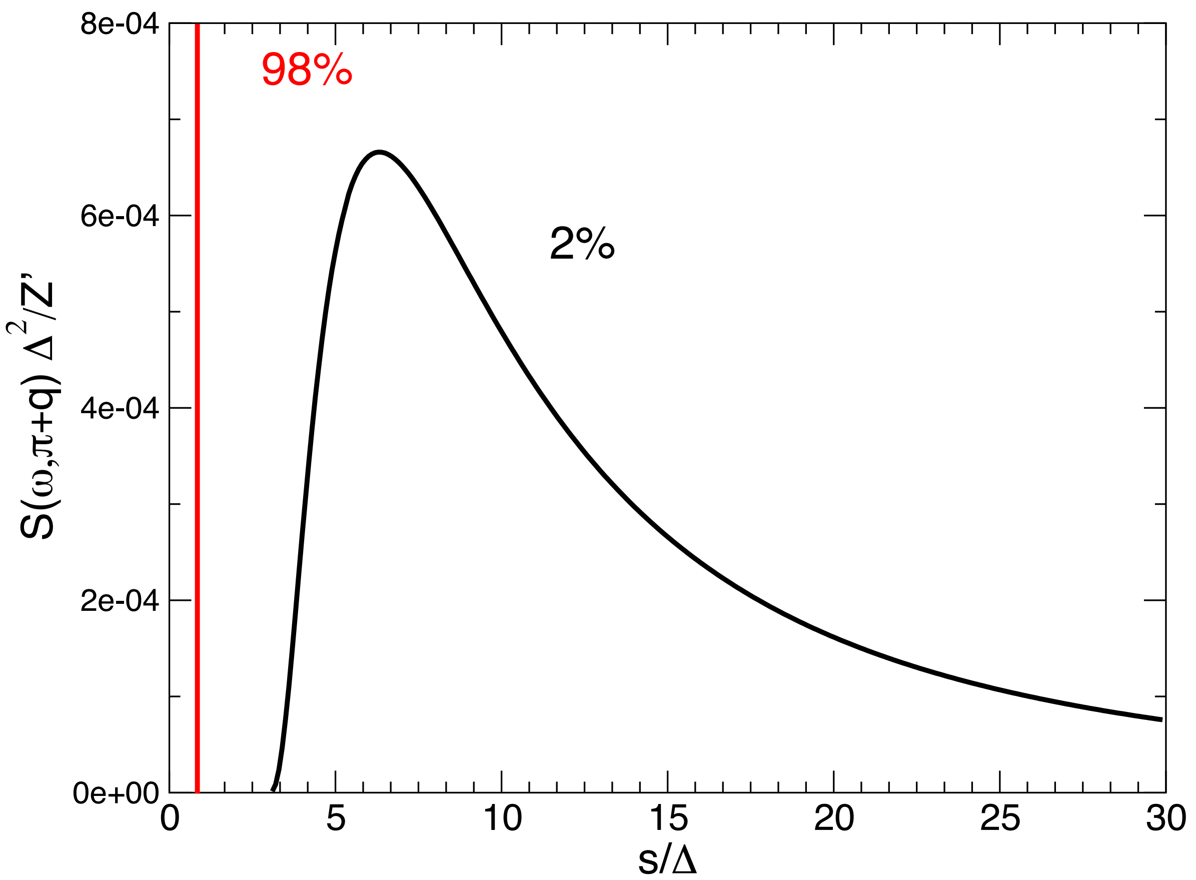
$$S(\omega, k = \frac{\pi}{a_0} + q) \propto \int dt\, dx\, e^{i\omega t - iqx} \langle 0|n^a(t,x)n^a(0,0)|0\rangle$$

$$\propto (2\pi)^2 \sum_m \Big|\langle 0|n^a(0,0)|m\rangle\Big|^2 \delta(q - P_m)\, \delta(\omega - E_m)$$

for $\omega < n\Delta$ at most n-1 part. states contribute ⇒ exact results.

T=0 DSF for O(3) nonlinear sigma model at q≈π

(Balog and Niedermaier '97, Affleck and Horton '99, Essler '00)



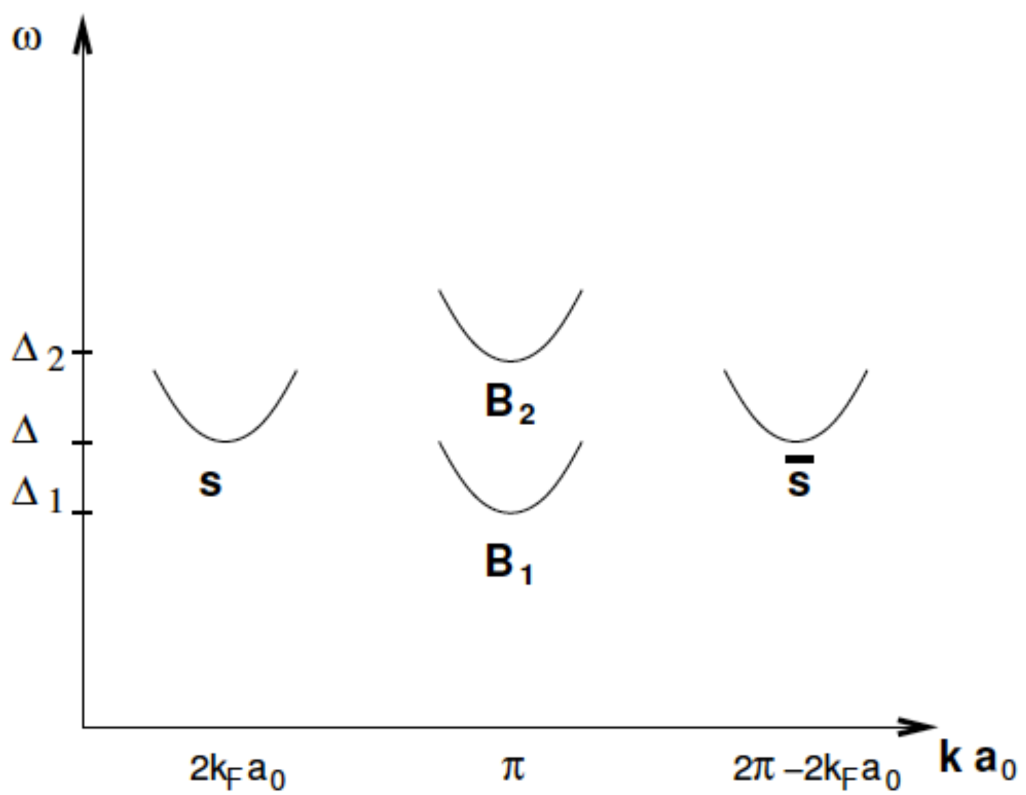
→ **very little** spectral weight above the magnon peak.

sine-Gordon model & field-induced gap systems

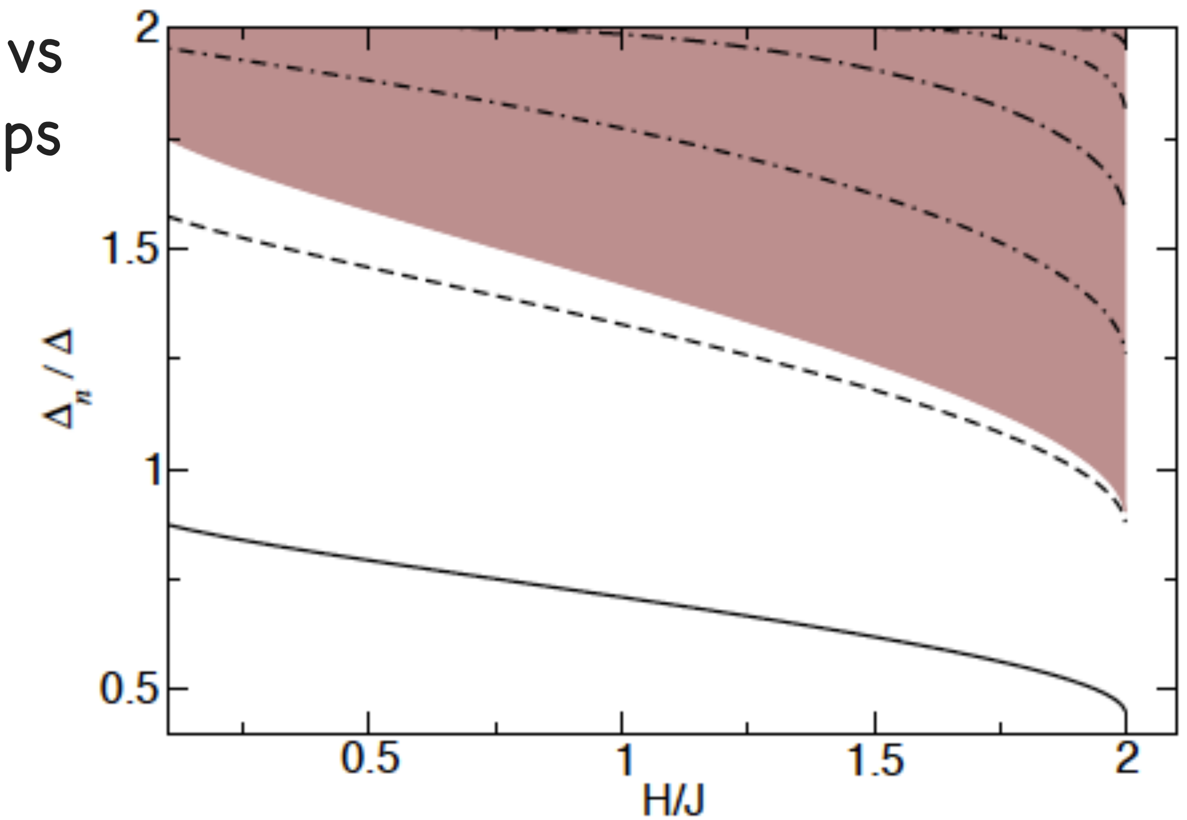
$$\mathcal{H} = \sum_j J\mathbf{S}_j \cdot \mathbf{S}_{j+1} - HS_j^z - h(-1)^j S_j^x \ .$$

where h=const H

elementary excitations: soliton, antisoliton and several(H) breathers

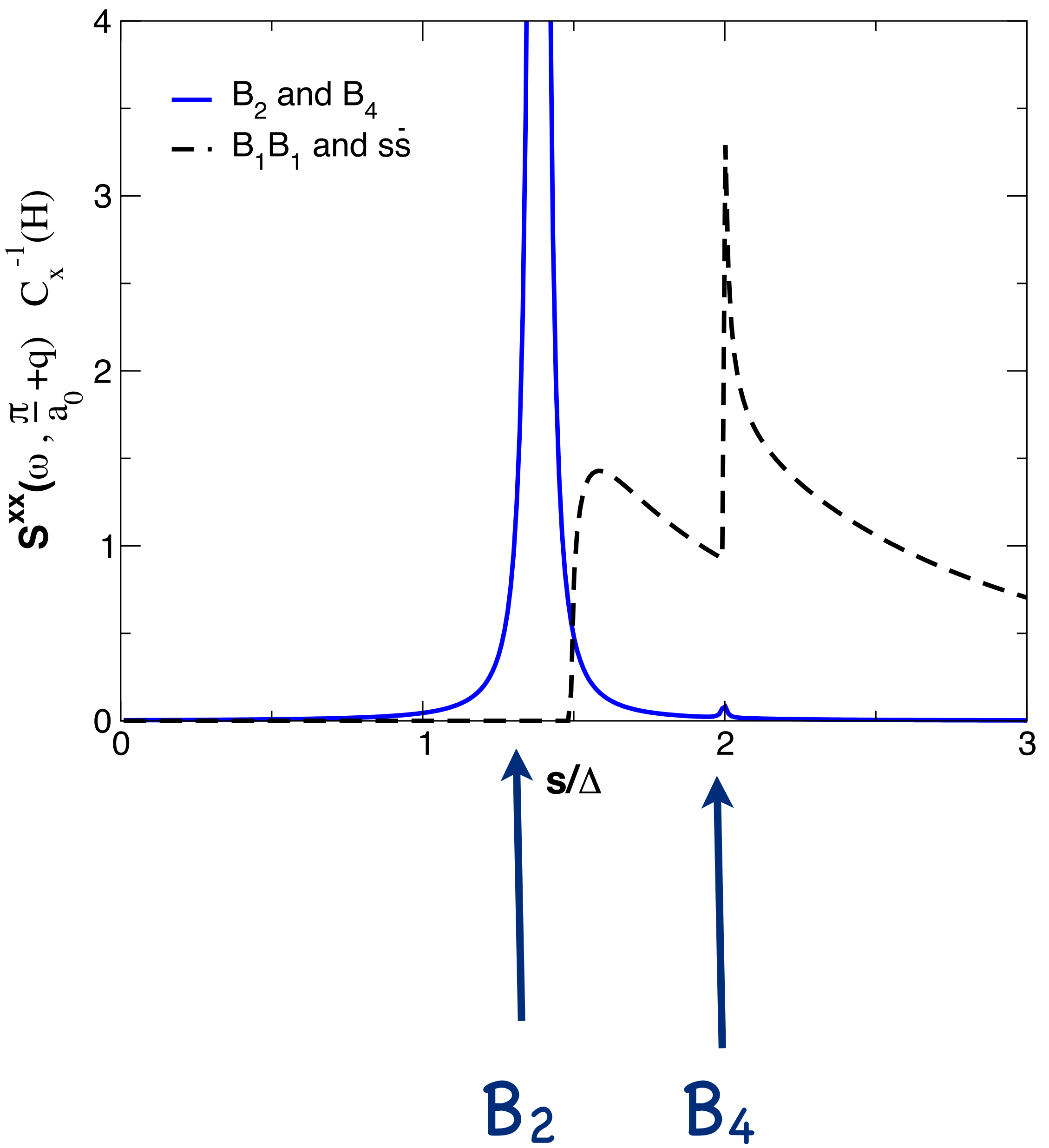


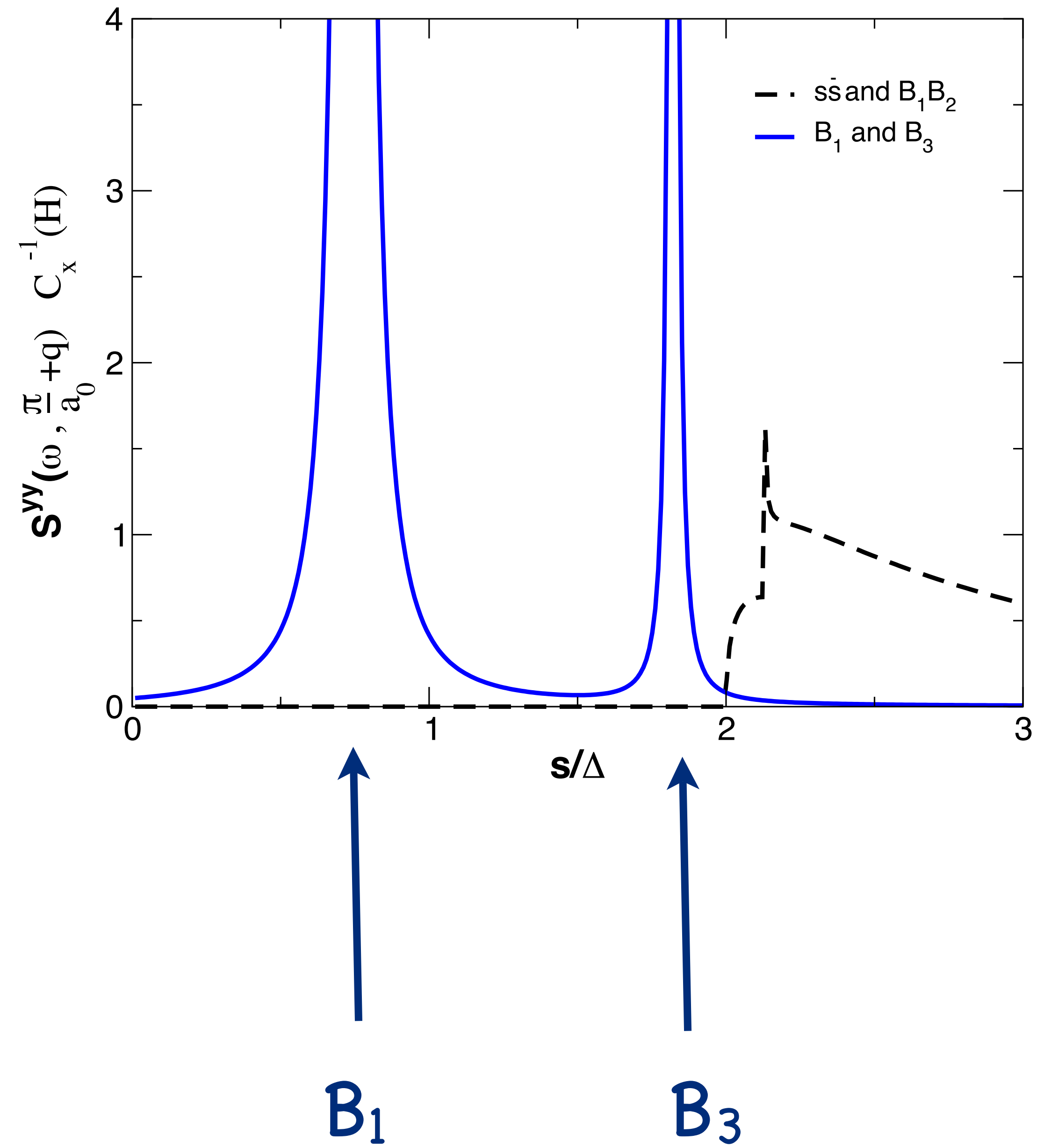
breather vs soliton gaps



Calculated Dynamical Structure Factor Components

Essler et al '97, '03

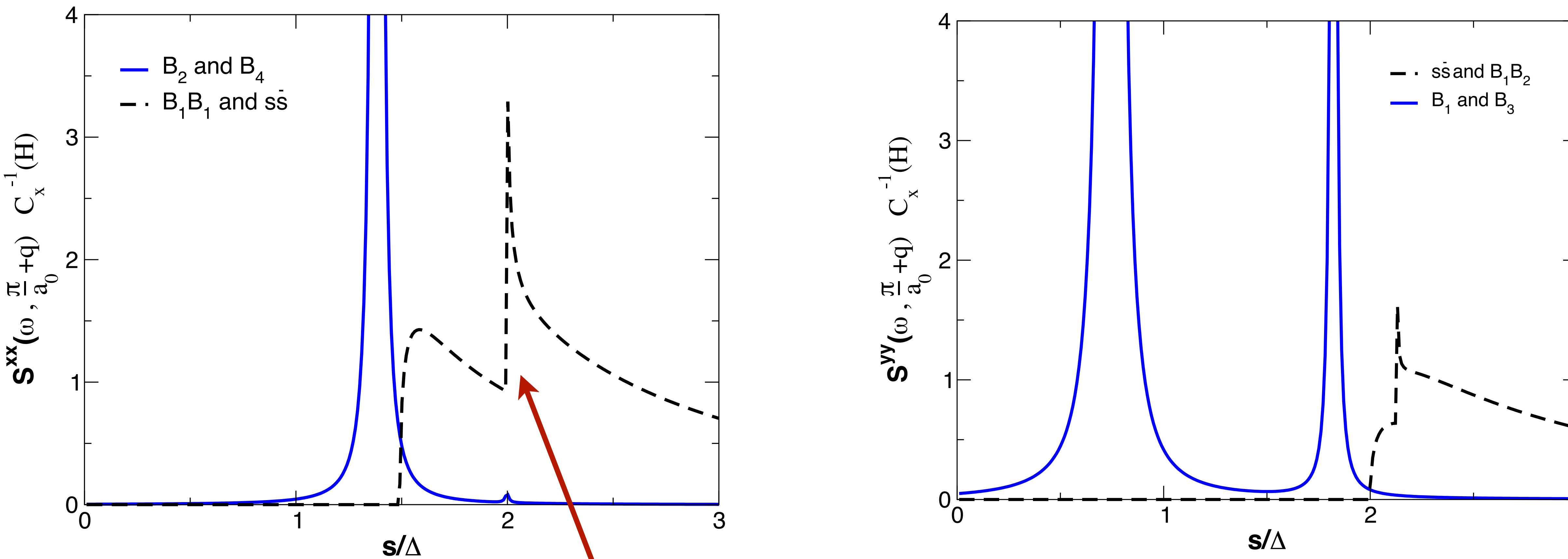




(delta-function breather peaks have been broadened by hand to show the spectral weight)

Calculated Dynamical Structure Factor Components

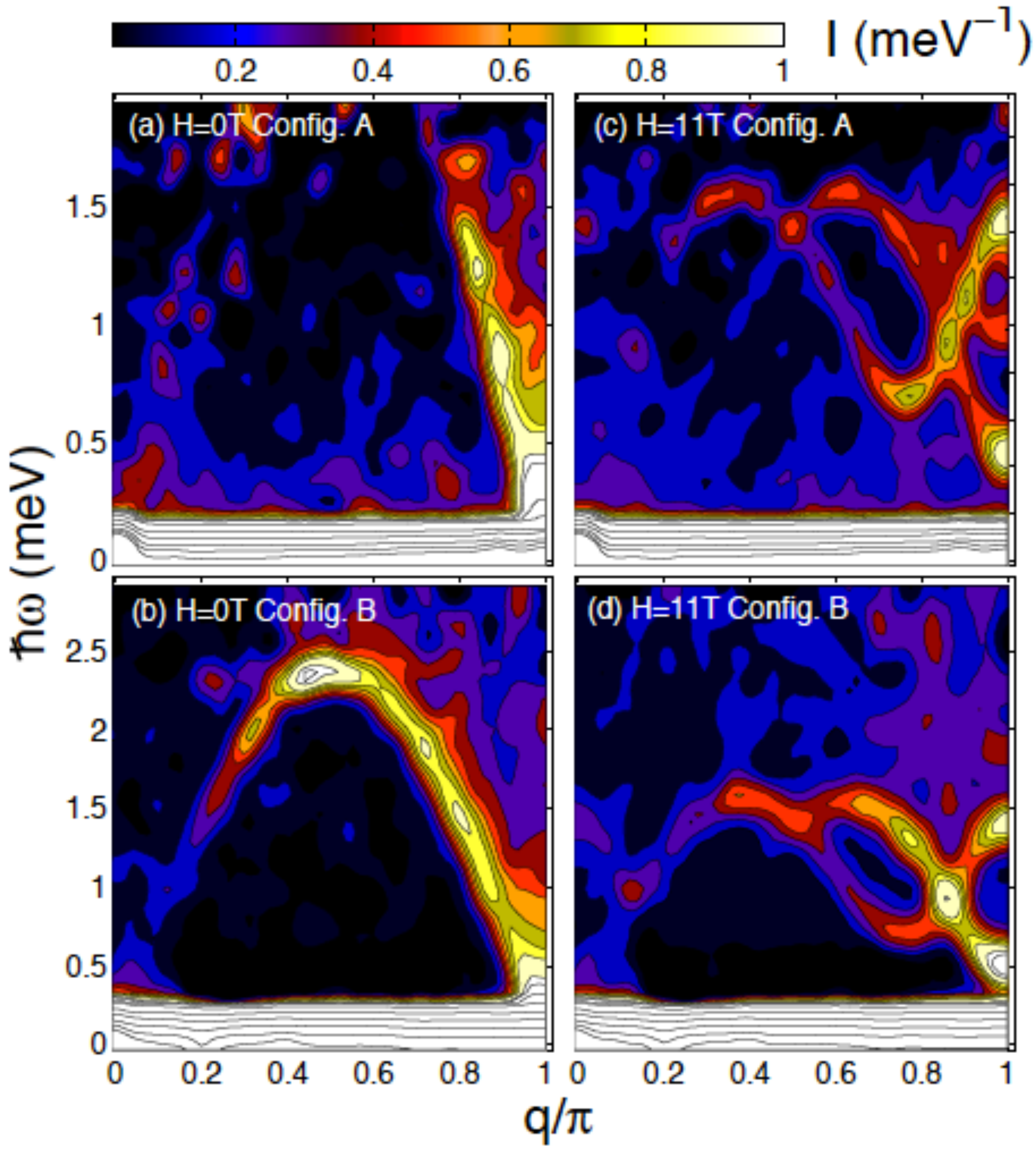
Essler et al '97, '03



threshold singularity on top of a scattering continuum.

Measured DSF for CDC

Kenzelmann et al '04



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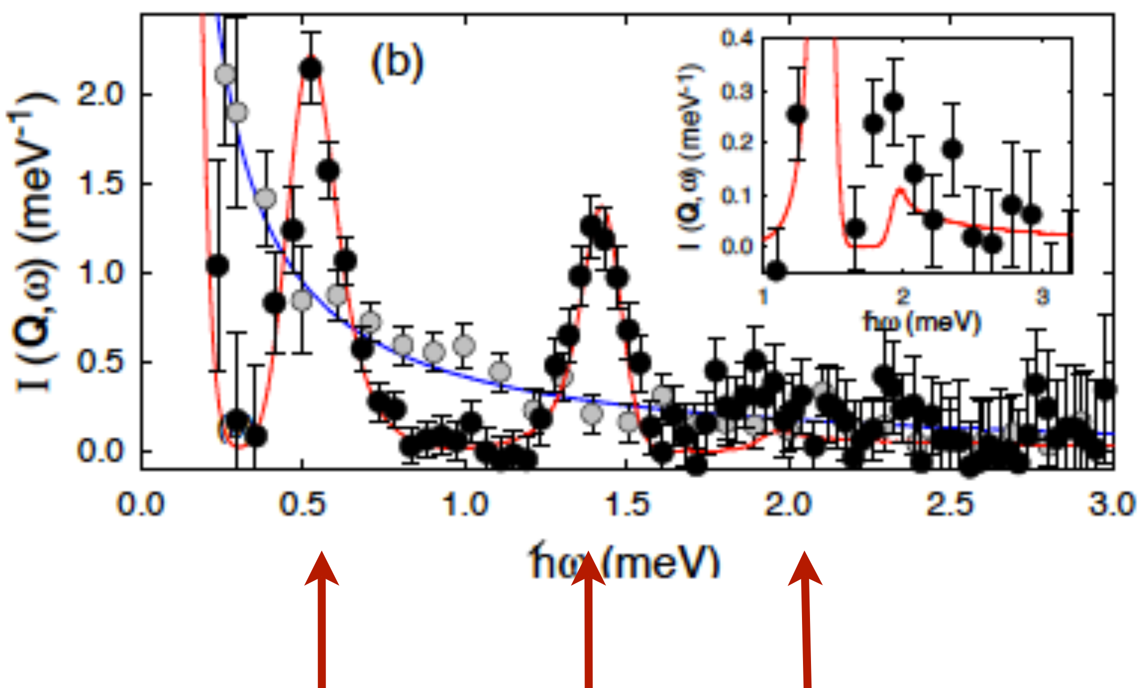
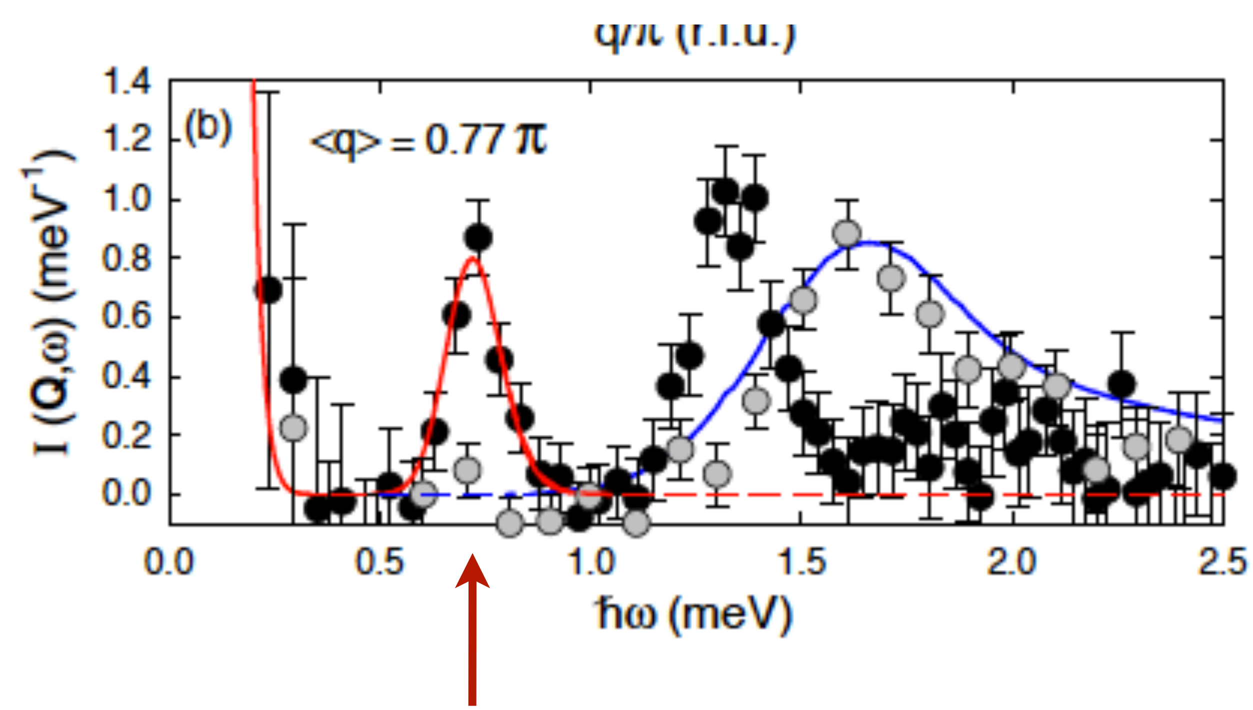


B_1 B_2 B_3



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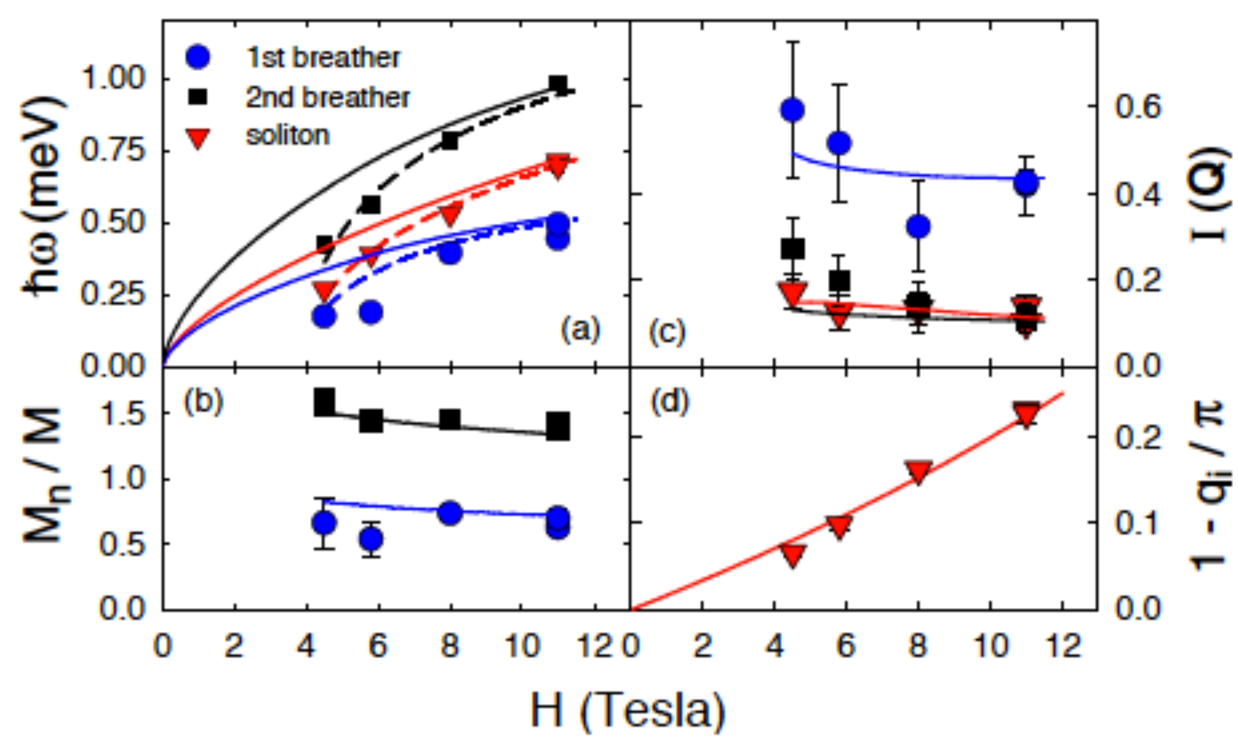

All of this was for $T=0$ - how about $T>0$?

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T>0 INS Studies of Gapped Quantum Spin Chains

A. Integer Spin Heisenberg Chains (Xu et al '00, '07, Kenzelmann et al '01)

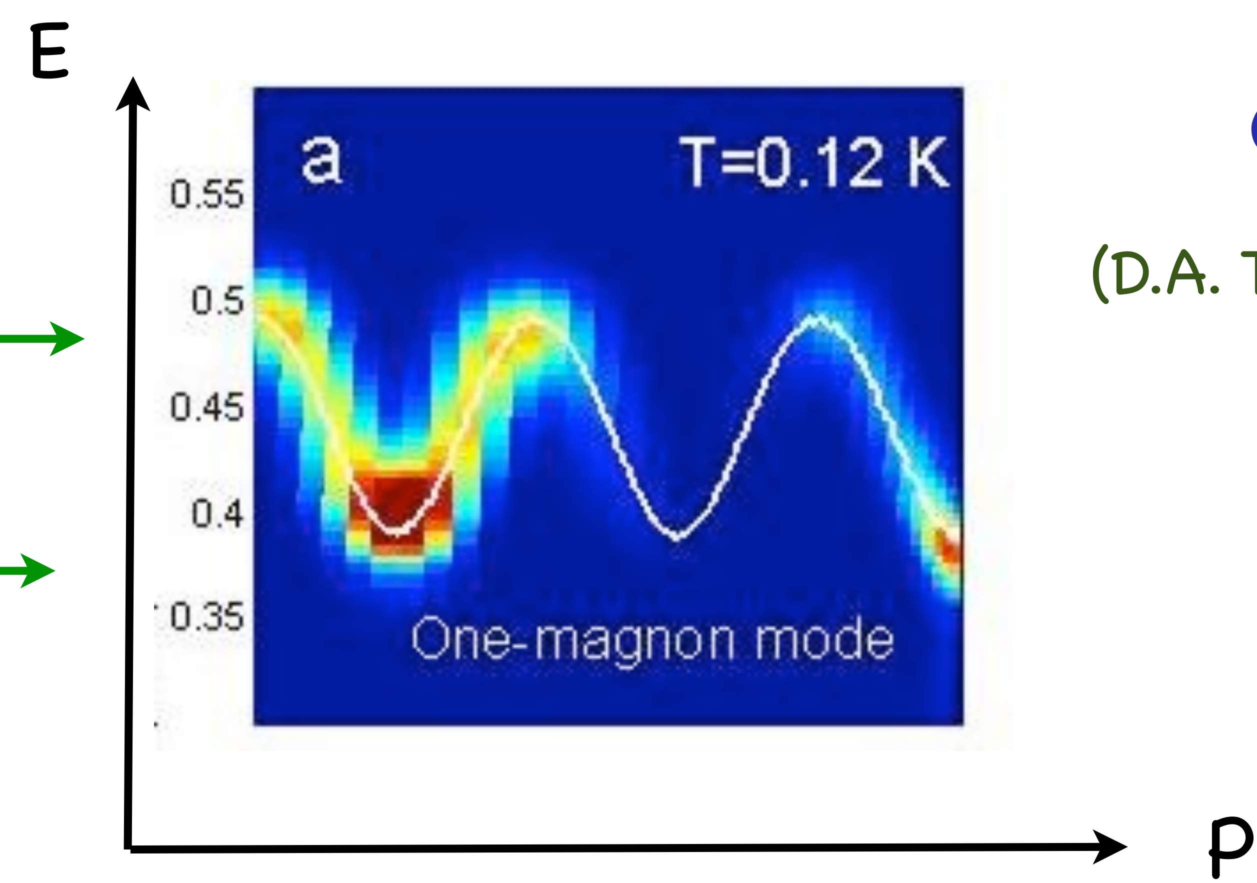
B. 2-Leg Heisenberg Spin-1/2 Ladders (Zheludev et al '08, Ruegg et al '10)

C. Dimerized Heisenberg Spin-1/2 Chains (Xu et al '00, Tennant et al '09)

T=0: spin singlet ground state & gapped triplet of coherent "magnon" excitations

"δ-function" arising from single particle

Gap



CuNitrate

(D.A. Tennant et al '09)

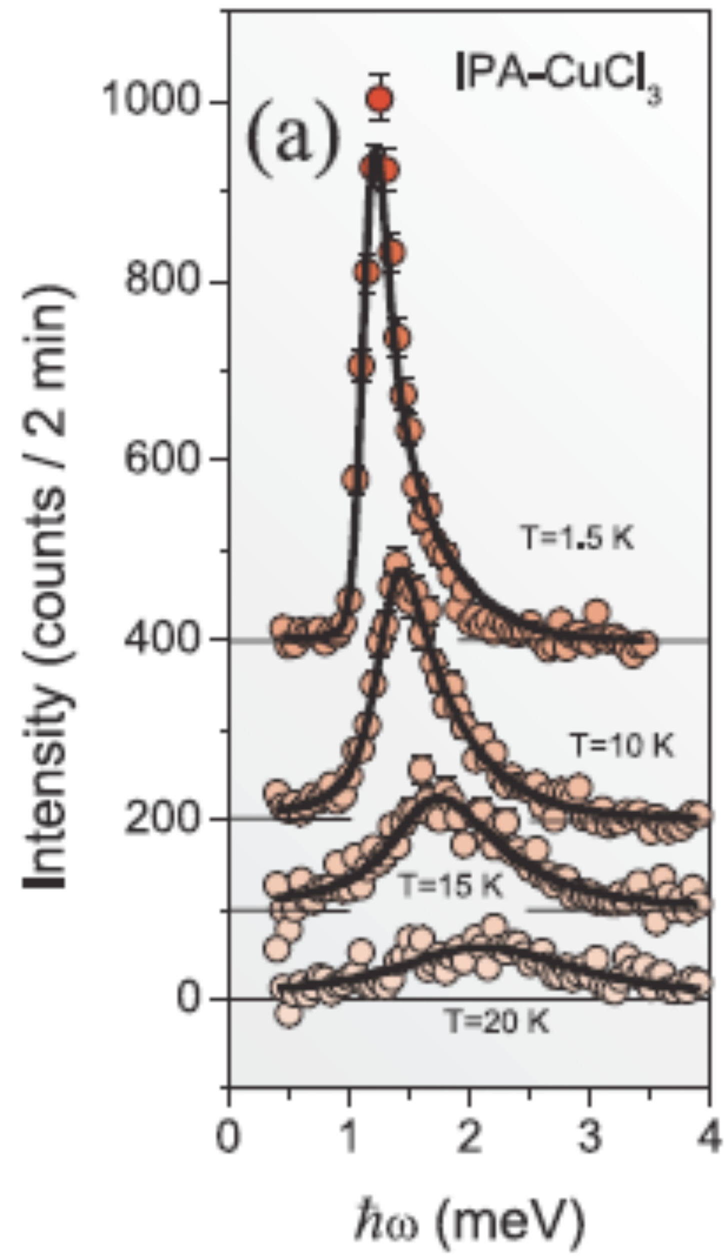
Linewidth and "T-dependent gap"

(A. Zheludev et al '08)

2-leg spin-1/2 ladder

(A) Line broadens with increasing T.

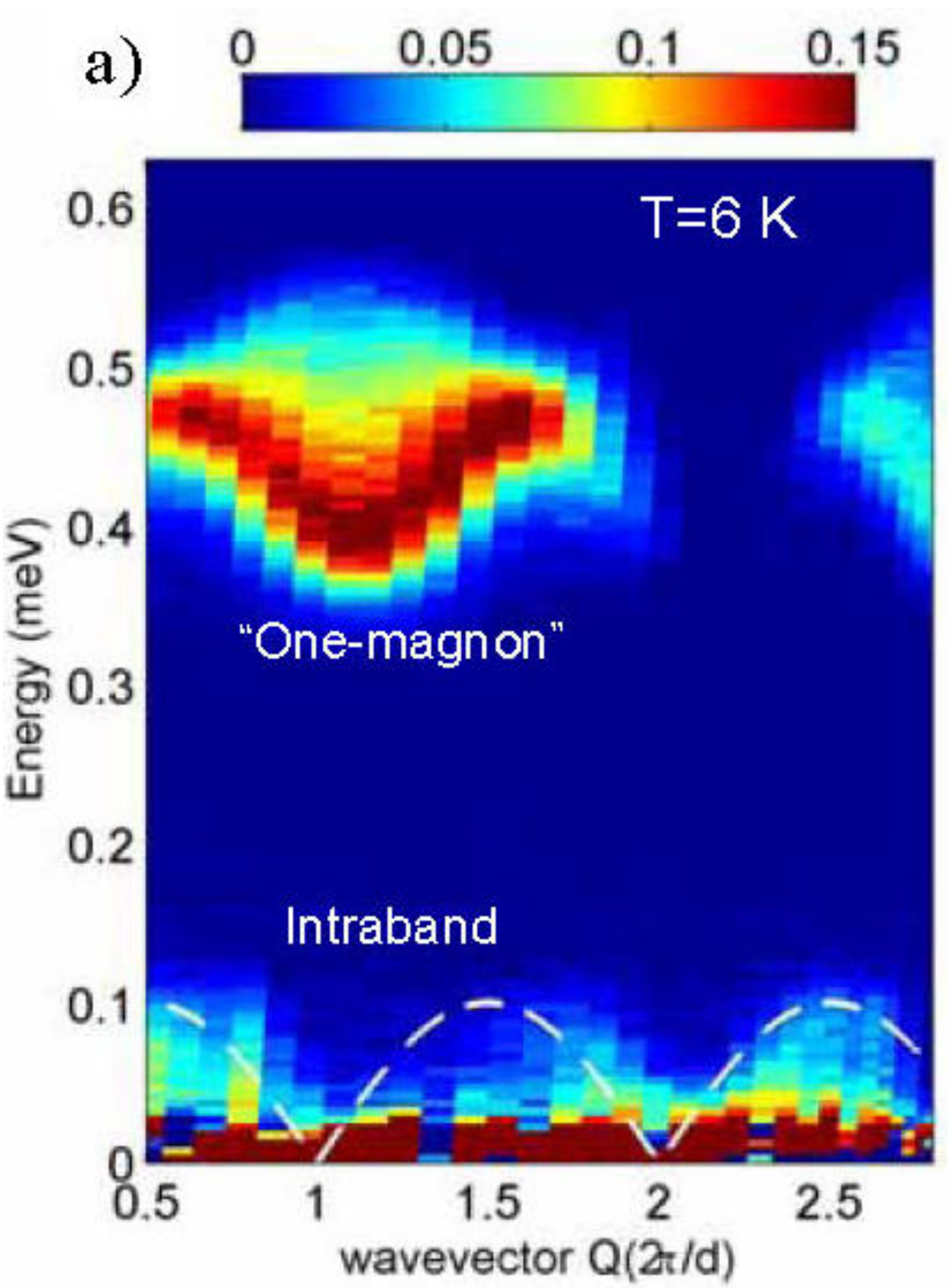
(B) Maximum shifts in energy: "T-dependent gap".



Thermally Activated Scattering

CuNitrate

$$H = \sum_j J\mathbf{S}_{2j} \cdot \mathbf{S}_{2j+1} + J'\mathbf{S}_{2j+1} \cdot \mathbf{S}_{2j+2}\ ,\ \ J' \ll J.$$



(D.A. Tennant et al '09, '12)

asymmetric broadening of magnon line

Thermally activated scattering

magnon already present in state of thermal equilibrium changes its spin

Why T>0 dynamics is a difficult problem

Dynamical Structure Factor

$$S(\omega, q) = \int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{i\omega t} \sum_l e^{-ikl} \langle S^a_{j+l}(t)\ S^a_j \rangle$$

T=0:

$$S(\omega, q) = 2\pi \sum_{\text{states } k} |\langle 0 | S^a_j | k \rangle|^2\ \delta(\omega - E_k + E_0)\ \delta(q - P_k)$$

Only need ground state and 1 magnon excitation

T>0:

$$S(\omega, q) = \frac{2\pi}{\mathrm{tr}(e^{-\beta H})} \sum_{\text{states } n,k} e^{-\beta E_n} |\langle n | S^a_j | k \rangle|^2\ \delta(\omega - E_k + E_n)\ \delta(q - P_k + P_n)$$

Have to deal with a **finite density** of magnons !

Lineshape of weakly interacting gapped fermions

- Single-particle spectral function of non-interacting fermions at T>0

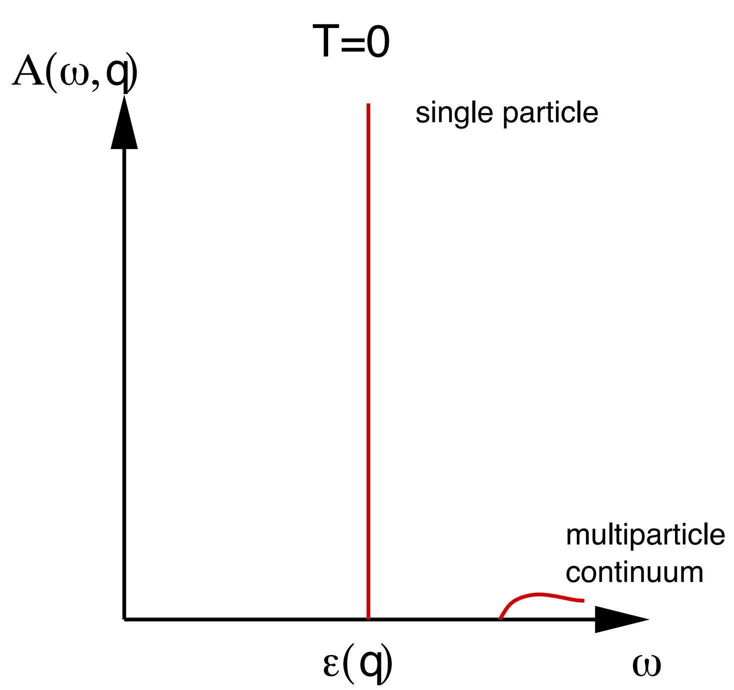
$$A(\omega,q) = -\frac{1}{\pi}\mathrm{Im}G_{\mathrm{ret}}(\omega>0,q) \propto \delta(\omega-\epsilon(q))$$

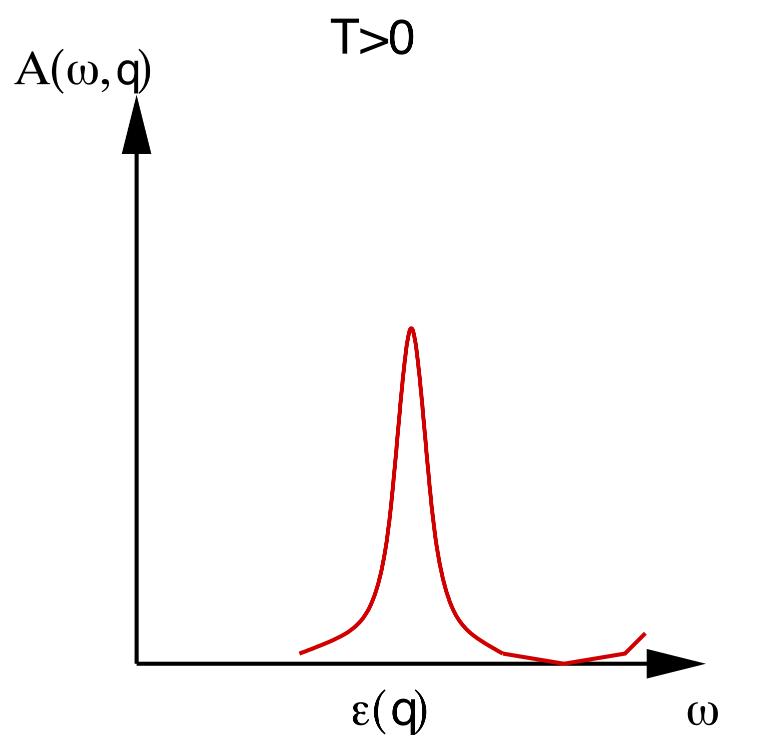
→**T-independent**. Need **interactions** to get non-trivial lineshape.

- Dynamical Response of weakly interacting, massive particles

$$A(\omega,q) \propto \mathrm{Im}\frac{1}{\omega+i0-\epsilon(q)+\Sigma(\omega,q,T)}$$

Σ(ω,q,T)calculable in (Matsubara) PT for **low T**.





Line-broadening gives fingerprint of interactions between particles.

Semiclassical Approach

(Sachdev, Damle '98)

(Rapp&Zarand '06,'09)

Consider gapped 1D quantum magnet with coherent single-particle excitation at T=0, dispersion $\varepsilon(q)$, gap $\Delta = \varepsilon(Q)$

(e.g. Haldane-gap chains, 2-leg ladder, transverse-field Ising model)

DSF $\quad S(\omega,q,T=0) \sim A(q)\,\delta(\omega-\varepsilon(q))$ + multi-particle

For "low T" the delta-function broadens in a **universal Lorentzian way** (at $q \approx Q$) as

$$S(\omega,q,T>0) \sim A(q)\,\tau^{-1}(T)\,[(\omega-\varepsilon(q))^2+\tau^{-2}(T)]^{-1} + \text{multi-particle}$$

widely used to analyze neutron data

Questions: Regime of applicability? How to go beyond?

Dynamical Correlations at T>0

(LeClair et al '96
LeClair/Mussardo '99
Konik '03 .....)

Basis of Hamiltonian eigenstates: $H|r\rangle = E_r|r\rangle$

$$\chi(\omega,q) = \frac{1}{\mathrm{tr}\,(e^{-\beta H})}\sum_{r,s}|\langle r|S^a(0,0)|s\rangle|^2\frac{e^{-\beta E_r}-e^{-\beta E_s}}{\omega+i\delta-E_s+E_r}2\pi\delta(q+P_r-P_s)$$

Magnon Gap $\Rightarrow$ $E_r \approx r\Delta$

Dynamical Correlations at T>0

Basis of Hamiltonian eigenstates: $H\,|r\rangle = E_r\,|r\rangle$

$$\chi(\omega,q) = \frac{1}{\mathrm{tr}\,(e^{-\beta H})}\sum_{r,s}|\langle r|S^a(0,0)|s\rangle|^2\frac{e^{-\beta E_r}-e^{-\beta E_s}}{\omega+i\delta-E_s+E_r}2\pi\delta(q+P_r-P_s)$$

Magnon Gap $\Rightarrow E_r \approx r\Delta$

Must be regularized!

(1) Finite Volume Regularization (Essler&Konik '08, Pozsgay& Takacs '08)

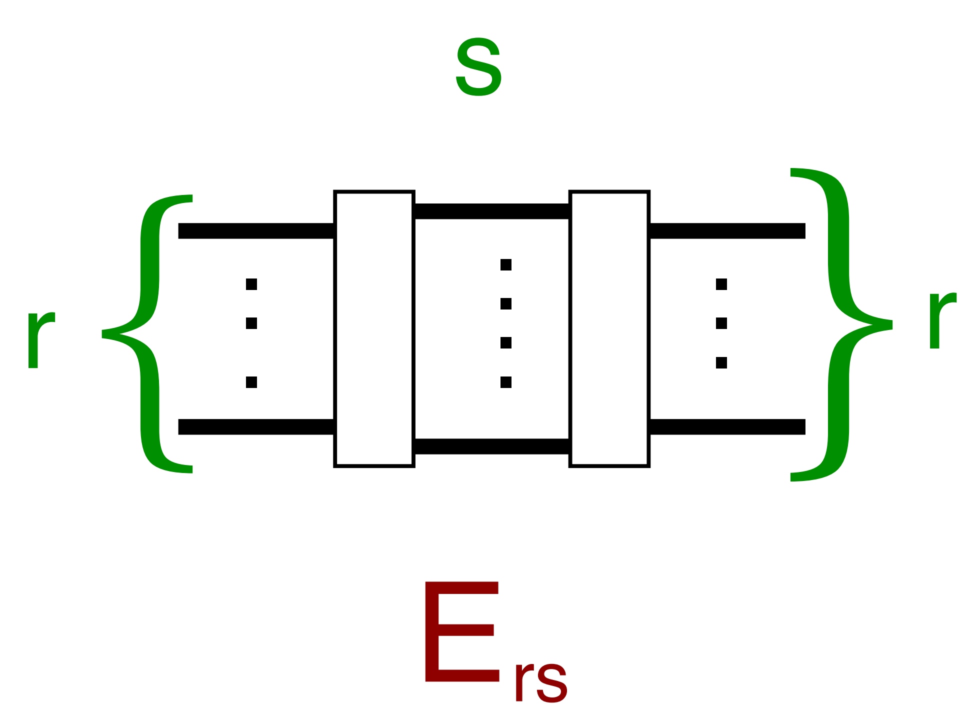
(2) Infinite Volume Regularization (Essler& Konik '09)

Dynamical Correlations at T>0

$$\chi(\omega,q) = \frac{1}{\mathrm{tr}\,(e^{-\beta H})}\sum_{r,s}|\langle r|S^a(0,0)|s\rangle|^2\,\frac{e^{-\beta E_r}-e^{-\beta E_s}}{\omega+i\delta-E_s+E_r}2\pi\delta(q+P_r-P_s)$$

$$\equiv \frac{1}{\mathrm{tr}\,(e^{-\beta H})}\sum_{r,s}\mathcal{E}_{r,s}(\omega,q)+\mathcal{F}_{r,s}(\omega,q).$$

Formally we have: $\mathcal{E}_{r,s}(\omega,q) = \mathcal{O}(e^{-r\beta\Delta})$



"Bare clusters"

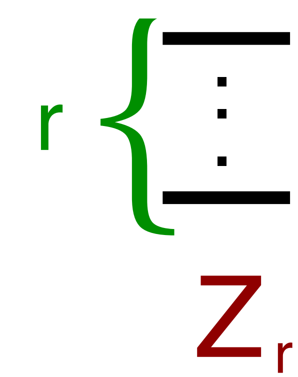
Dynamical Correlations at T>0

$$
\begin{aligned}
\chi(\omega,q) \quad &= \quad \frac{1}{\mathrm{tr}\left(e^{-\beta H}\right)} \sum_{r,s} |\langle r|S^a(0,0)|s\rangle|^2 \frac{e^{-\beta E_r} - e^{-\beta E_s}}{\omega + i\delta - E_s + E_r} 2\pi\delta(q + P_r - P_s) \\
&\equiv \quad \frac{1}{\mathrm{tr}\left(e^{-\beta H}\right)} \sum_{r,s} \mathcal{E}_{r,s}(\omega,q) + \mathcal{F}_{r,s}(\omega,q).
\end{aligned}
$$

Formally we have: $\mathscr{F}_{r,s}(\omega,q) = \mathcal{O}(e^{-s\beta\Delta})$

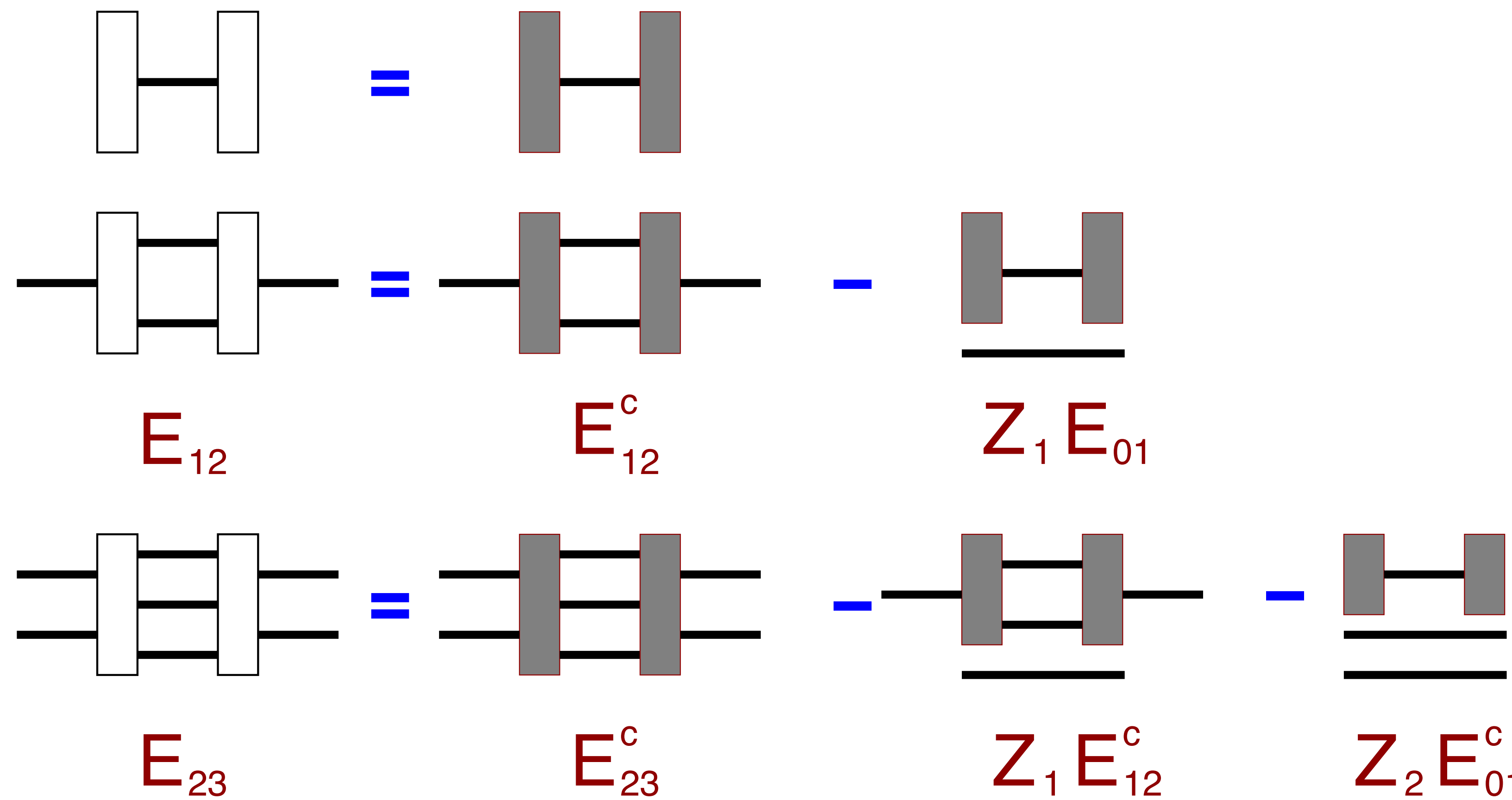
Partition Function:

$$
\mathrm{tr}\left(e^{-\beta H}\right) = \sum_m e^{-\beta E_l}\langle l|l\rangle \equiv 1 + \sum_{n=1}^{\infty} Z_n \,, \qquad Z_n = \mathcal{O}(e^{-n\beta\Delta}).
$$

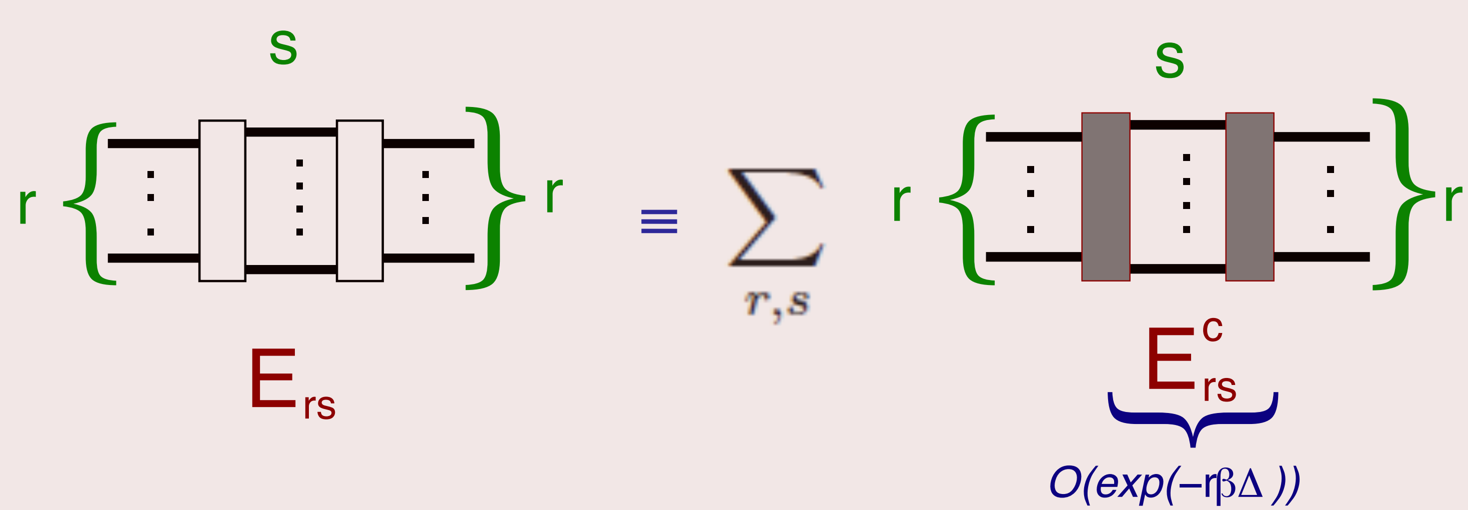


Low-Temperature Expansion

Define **"linked clusters"**:



$$\chi_{\varepsilon}(\omega, q) = \frac{1}{\mathrm{tr}\,(e^{-\beta H})} \sum_{r,s}$$



low-T expansion for χ(ω,q):

$$\chi(\omega, q) = \underbrace{C_0}_{\mathcal{O}(1)} + \underbrace{C_1}_{\mathcal{O}(e^{-\beta\Delta})} + \underbrace{C_2}_{\mathcal{O}(e^{-2\beta\Delta})} + \ldots \quad \text{(LTE)}$$

Problem: subleading (in exp(-βΔ)) terms in χ(ω,q) more and more divergent when $\omega^2 \to \varepsilon^2(q)$

Solution:

(1) define Σ(ω,q) through $$\chi(\omega, q) = \frac{C_0(\omega, q)}{1 - C_0(\omega, q)\Sigma(\omega, q, T)} .$$

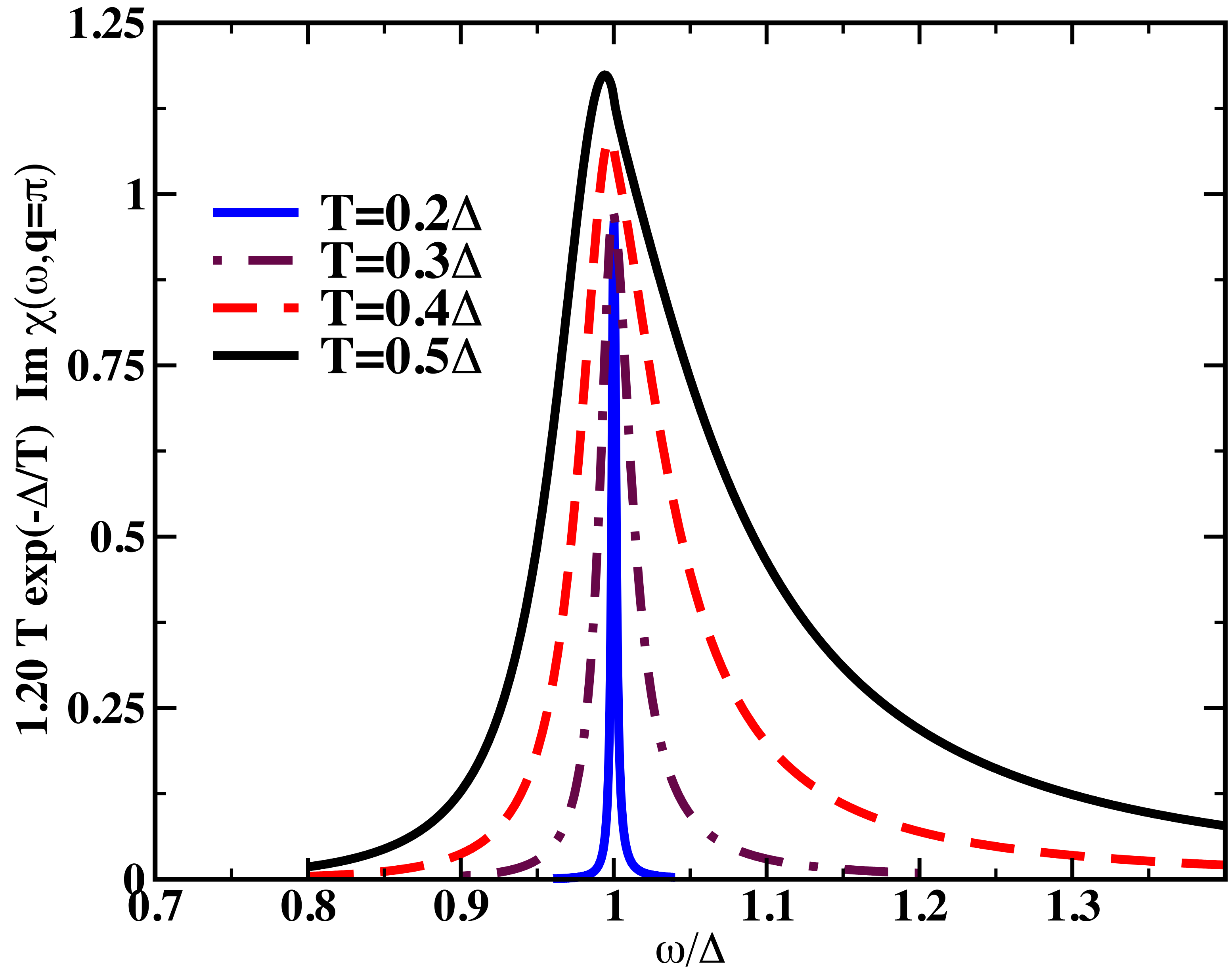
(2) Match expansion $$\chi(\omega, q) \approx C_0(\omega, q) + C_0^2(\omega, q)\Sigma(\omega, q, T) + \ldots$$

to LTE ⇒ Σ(ω,q,T)

Results for T>0 Lineshape in O(3) nlΣm

DSF at $q=\pi/a_0$ for integer spin-S Heisenberg chain at low energies

- width $\propto T \exp(-\beta\Delta)$
- height $\propto T^{-1} \exp(\beta\Delta)$
- lineshape **Lorentzian** for $T \ll \Delta$, $\omega \approx \Delta$ ⇒ agrees w. semiclassical approx. (Damle/Sachdev '98)
- lineshape **asymmetric** for any T and $\omega \neq \Delta$
- asymmetry increases w. T

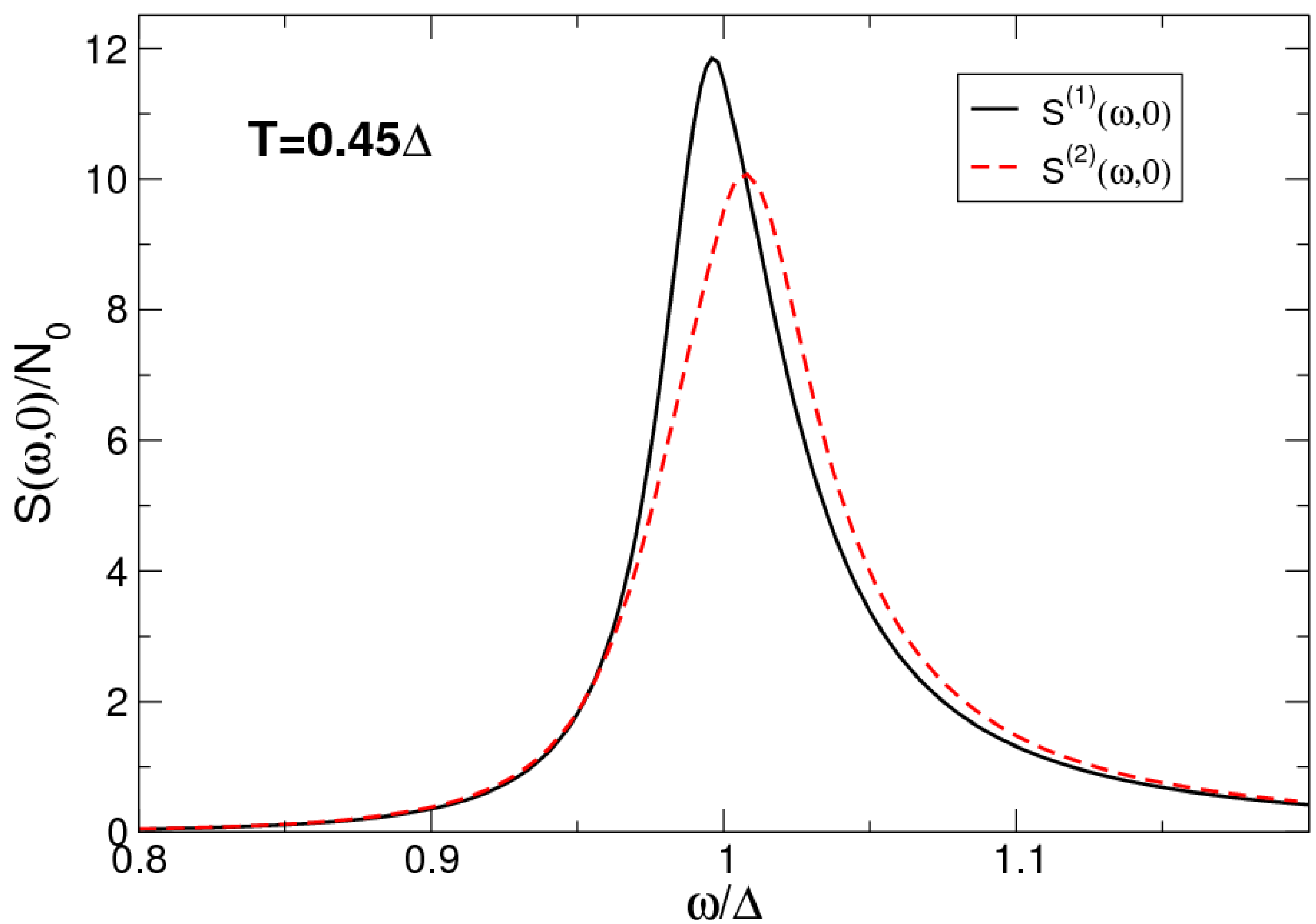


Damle/Sachdev result recovered at rather low T<0.1Δ

Higher Orders in Low-T Expansion

$$H_{\mathrm{TFIM}} = J \sum_n S_n^z S_{n+1}^z - h \sum_n S_n^x \,, \qquad 1 \gg \frac{h}{J} - 1 > 0$$

$S^{zz}(\omega, q=0, T)$



Maximum moves in energy

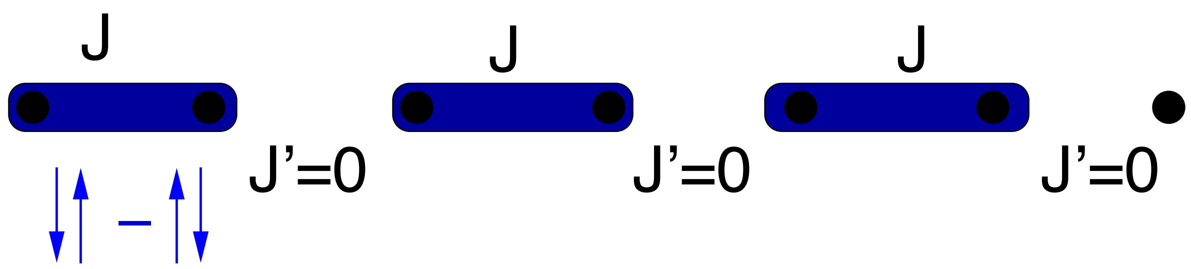
"T-dependent gap"

Non-Integrable Models

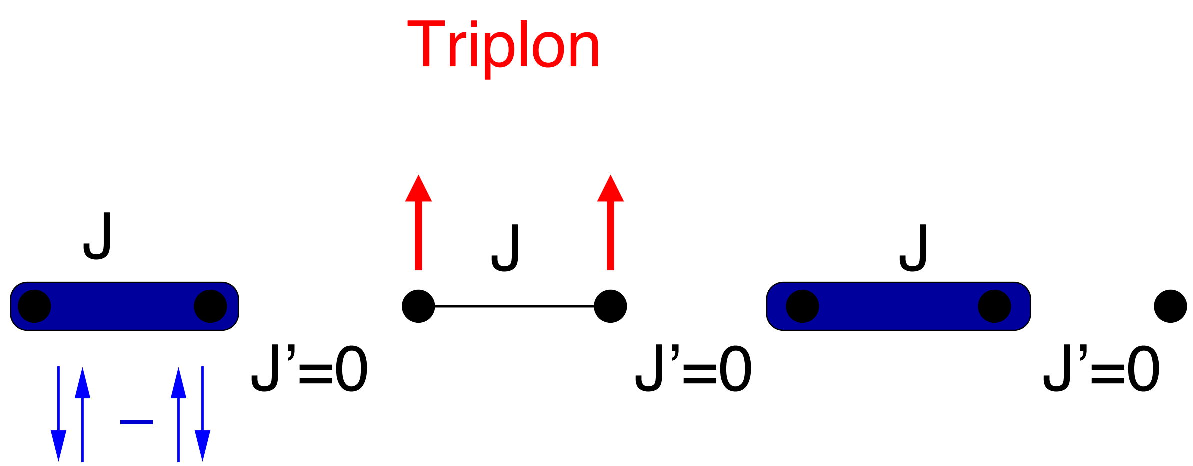
e.g. Alternating spin-½ Heisenberg Chain:

$$H = \sum_j J\mathbf{S}_{2j} \cdot \mathbf{S}_{2j+1} + J'\mathbf{S}_{2j+1} \cdot \mathbf{S}_{2j+2} \;,\;\; J' \ll J.$$

J′=0 limit: uncoupled dimers



Ground State



"Triplon" S=1 Excitation

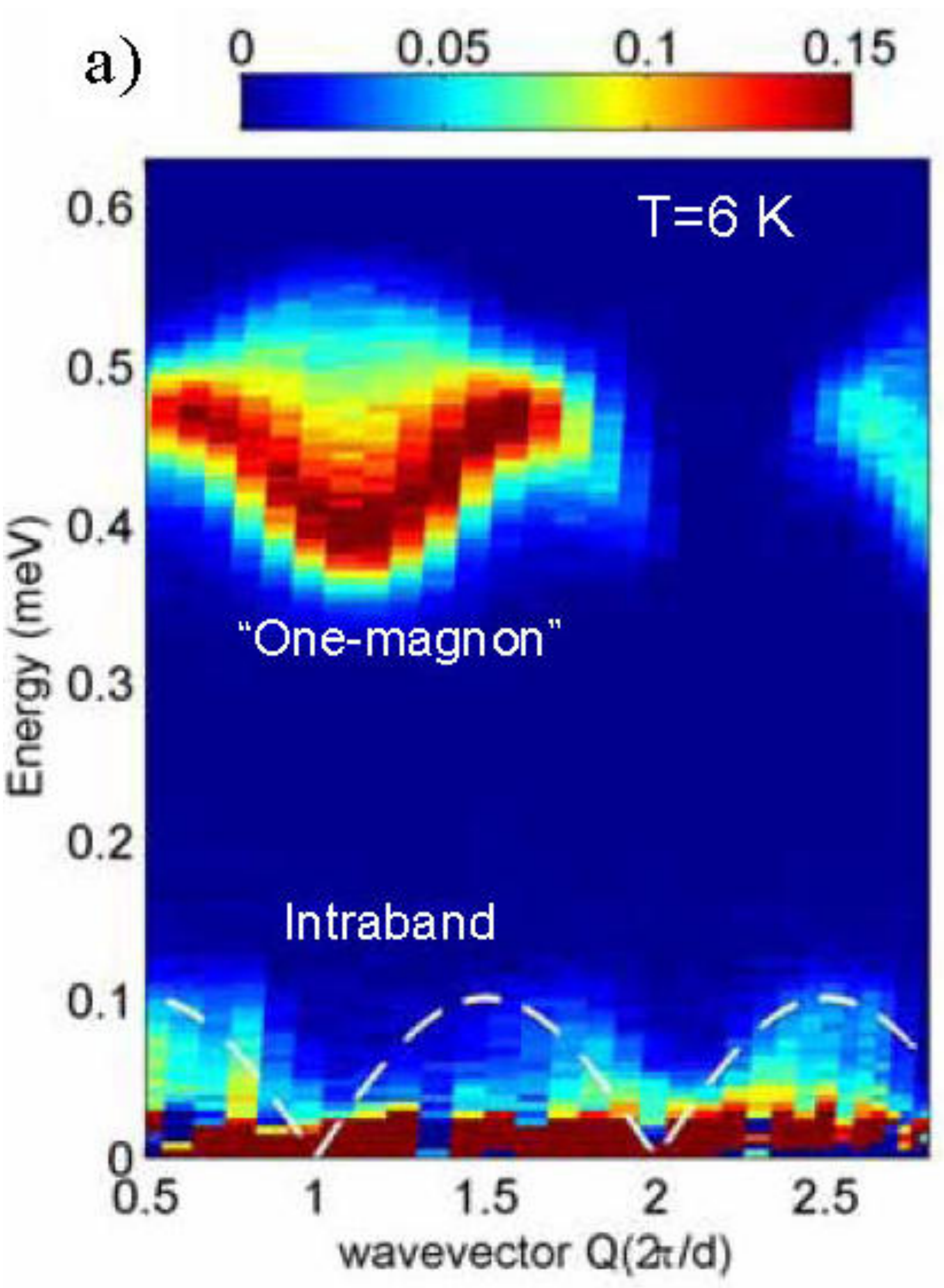
combine low-T expansion with Perturbation Theory in J′

Alternating spin-1/2 Heisenberg Chain CuNitrate

(D.A. Tennant et al '09)

$$H = \sum_j J\mathbf{S}_{2j} \cdot \mathbf{S}_{2j+1} + J'\mathbf{S}_{2j+1} \cdot \mathbf{S}_{2j+2} \;, \;\; J' \ll J.$$

J'≈0.23J



asymmetric broadening of magnon line

Thermally activated scattering

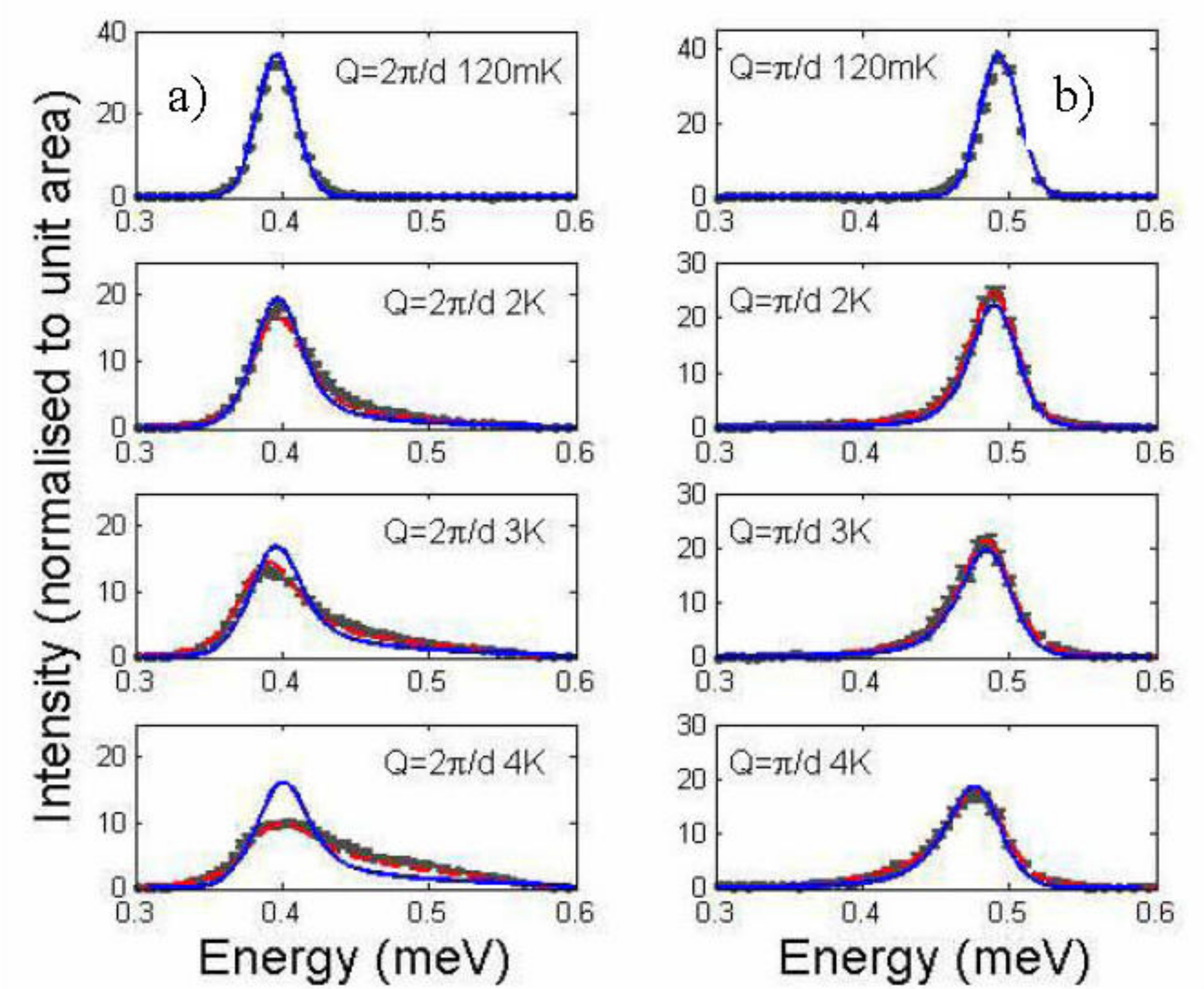
magnon already present in state of thermal equilibrium changes its spin

Experiments on CuNitrate

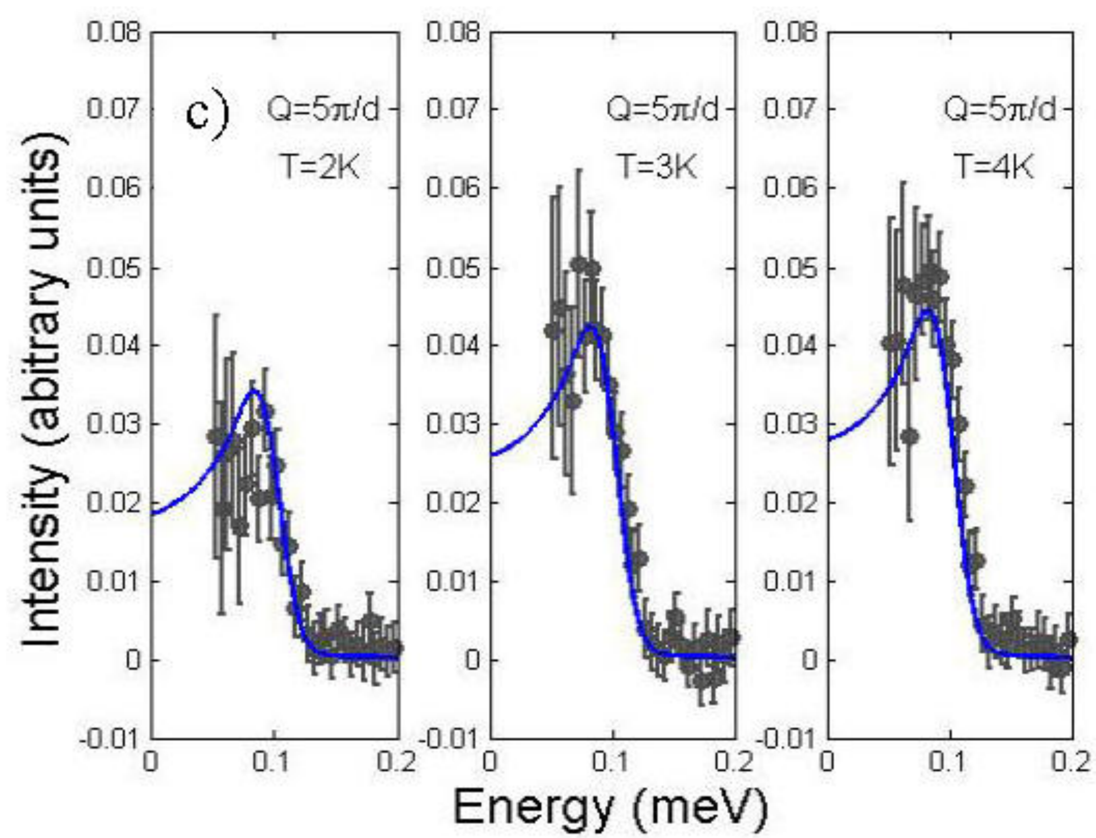
(D.A. Tennant et al)

Constant Q scans: S(ω,Q,T) for fixed Q and T

"Triplon" Broadening



low-energy T>0 Resonance



Summary

- Many gapped quantum spin chains reduce to integrable QFTs at low energies.
- There is an efficient method to calculate $T=0$ dynamical response functions in these QFTs.
- Low-T expansion for finite temperature dynamics.
- Low-T expansion applicable to certain non-integrable spin chains.
- Methods developed for $T>0$ dynamics allow analysis of non-equilibrium problems.

Outline

1. Why low-D Quantum Spin Systems are interesting.

2. Some Gapped Quantum Spin Chains and their Quantum Field Theory Limits.

3. Integrable QFTs.

4. T=0 Dynamical Response Functions in integrable QFTs.

5. T>0 Dynamical Response Functions (in integrable QFTs).

6. Nonequilibrium Dynamics.

Quantum systems out of equilibrium

Idea:

A. Consider a quantum many-particle system with Hamiltonian H

B. Prepare the system in a state $|\psi\rangle$ that is **not** an eigenstate.

C. Time evolution $|\psi(t)\rangle = \exp(-iHt)\,|\psi\rangle$

D. Study time evolution of local observables $\langle\psi(t)|O(x)|\psi(t)\rangle$ in the **thermodynamic limit**.

Quantum systems out of equilibrium

Idea:

A. Consider a quantum many-particle system with Hamiltonian H

B. Prepare the system in a state $|\psi\rangle$ that is **not** an eigenstate.

C. Time evolution $|\psi(t)\rangle = \exp(-iHt)\,|\psi\rangle$

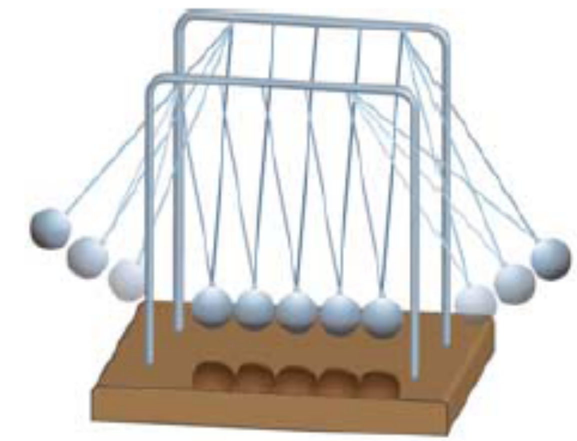
D. Study time evolution of local observables $\langle\psi(t)|O(x)|\psi(t)\rangle$ in the **thermodynamic limit**.

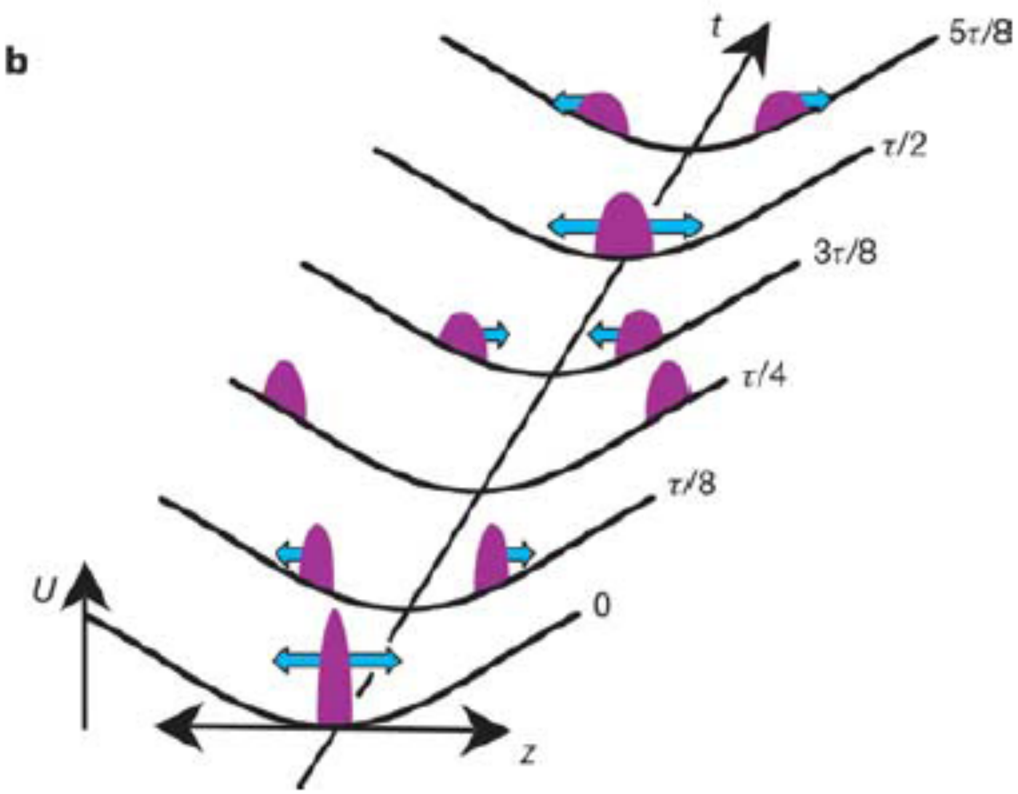
$|\psi(t)\rangle$ has a **finite density** of elementary excitations of the new Hamiltonian H → **like T>0**

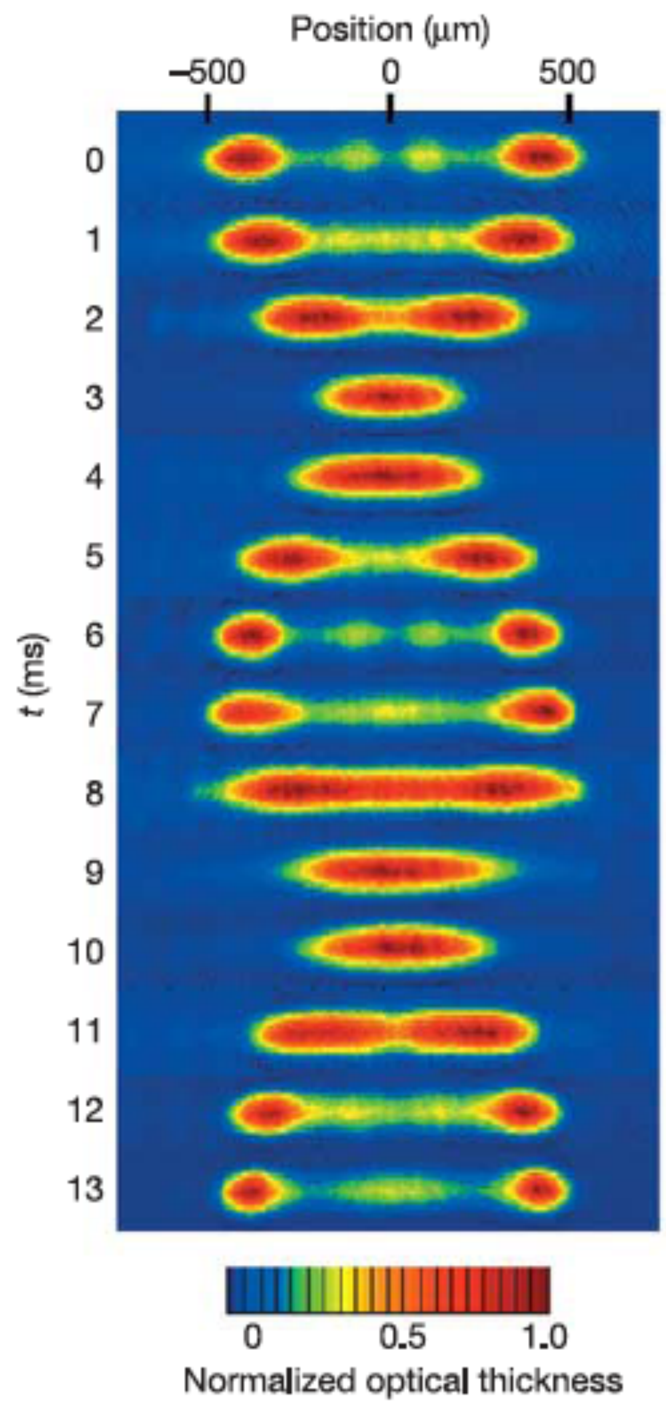
Experiments: "Quantum Newton's Cradle"

T. Kinoshita, T. Wenger and D.S. Weiss, Nature 440, 900 (2006)

40-250 ^{87}Rb atoms in a 1D optical trap







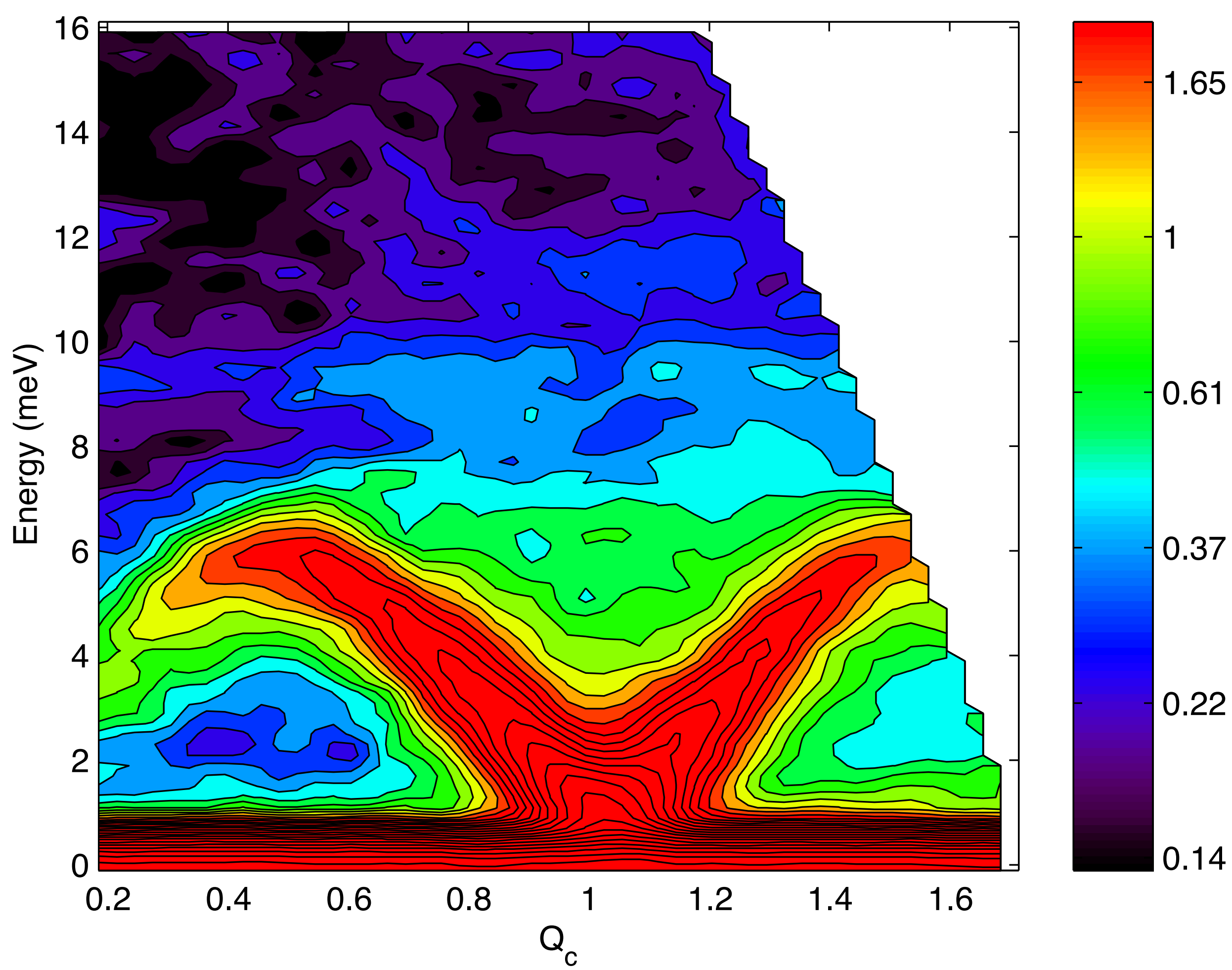
Essentially unitary time evolution.

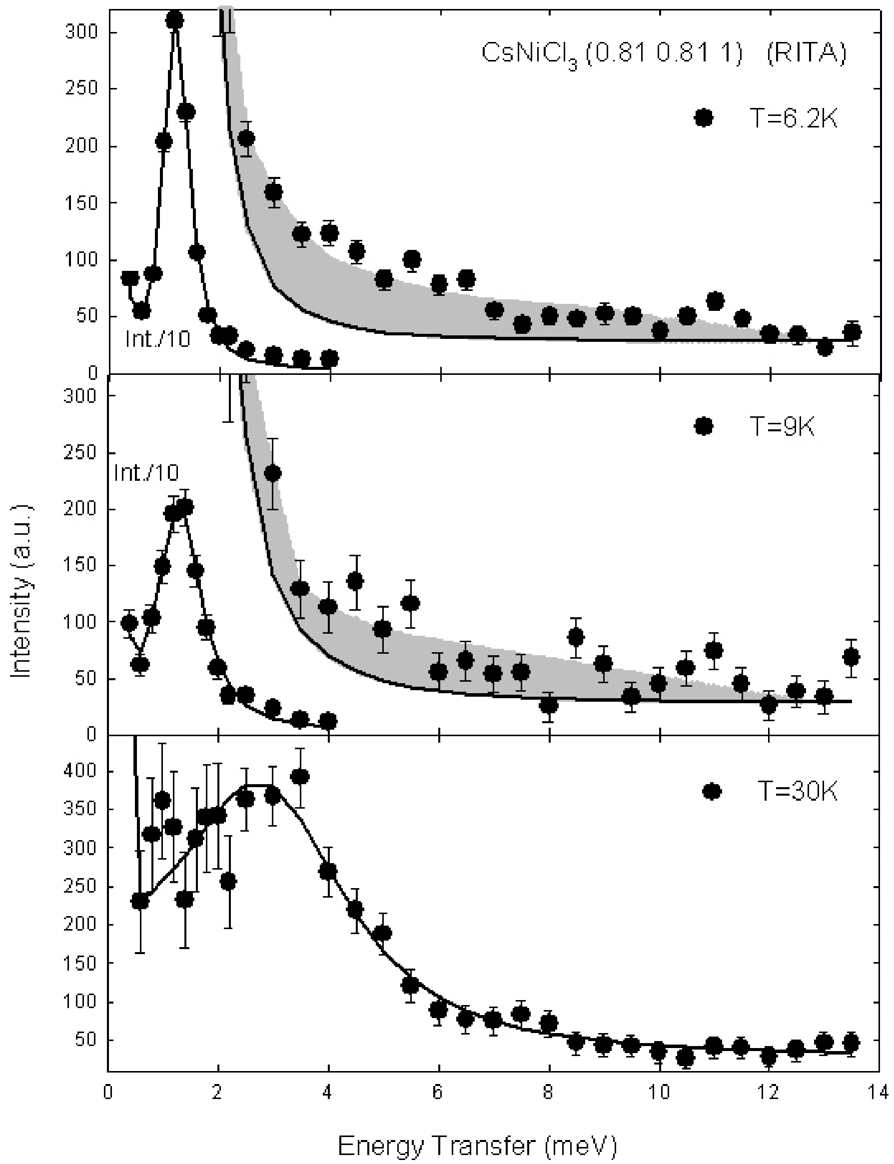
A different talk !

☺

T>0 Dynamical Structure Factor of $CsNiCl_3$

(M. Kenzelmann et al '01) (spin-1 Heisenberg chain)





T=0 delta-function → asymmetric continuum at T>0

Dynamical Structure Factor

Continuum limit of lattice spin operators:

$$\mathbf{S}_j \approx S(-1)^{ja_0}\mathbf{n}(x) + \mathbf{M}(x) \ , \qquad \mathbf{M}(x) = \frac{1}{vg}\mathbf{n}(x) \times \frac{\partial \mathbf{n}(x)}{\partial t}.$$

Dynamical structure factor around q = π:

$$S(\omega, k = \frac{\pi}{a_0} + q) \propto \int dt\ dx\ e^{i\omega t - iqx} e^{iq(R_j - R_l)} \langle n^a(t,x) n^a(0,0)\rangle$$

Dynamical structure factor around q = 0:

$$S(\omega, q) \propto \int dt\ dx\ e^{i\omega t - iqx} e^{iq(R_j - R_l)} \langle M^a(t,x) M^a(0,0)\rangle$$

Restrictions: formally S>>1, ω<< J (⇔ $|qa_0|$<< π).

Non-Integrable Models

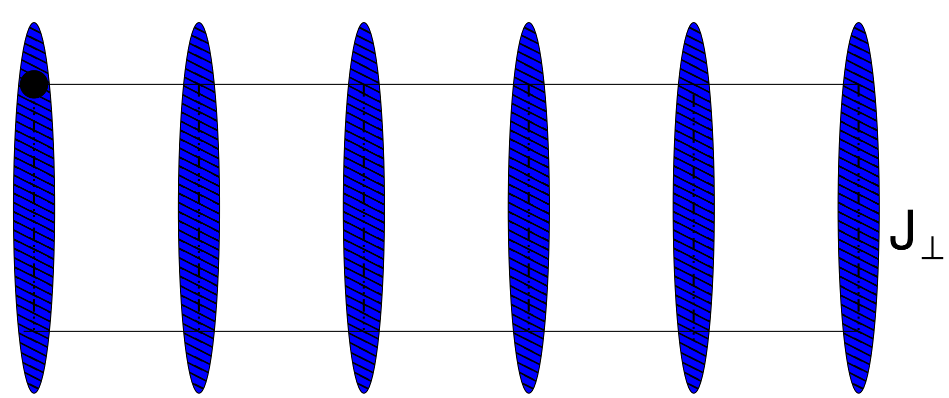
A. Alternating spin-½ Heisenberg Chain:

$$H = \sum_j J\mathbf{S}_{2j} \cdot \mathbf{S}_{2j+1} + J'\mathbf{S}_{2j+1} \cdot \mathbf{S}_{2j+2} \ , \ \ J' \ll J.$$

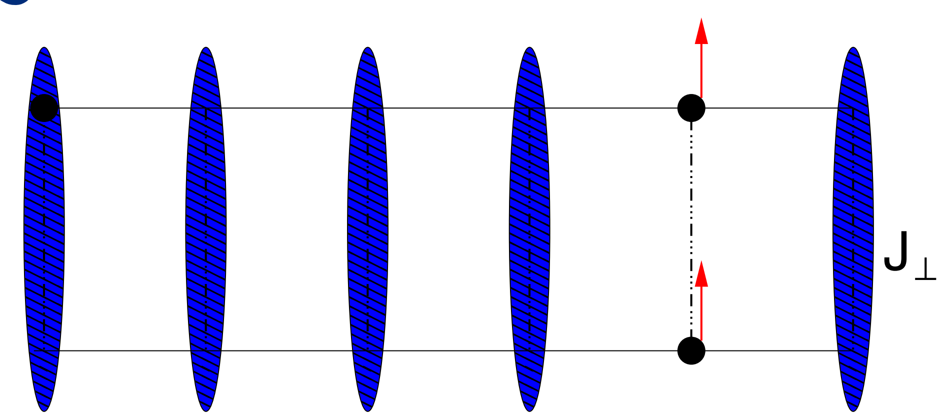
B. 2-leg spin-½ Heisenberg ladder:

$$H = J_{\parallel} \sum_{a=1,2} \sum_j \mathbf{S}_{a,j} \cdot \mathbf{S}_{a,j+1} + J_{\perp} \sum_j \mathbf{S}_{1,j} \cdot \mathbf{S}_{2,j} \ , \ \ J_{\parallel} \ll J_{\perp}.$$

$J_{||}$=0: uncoupled dimers (for A. along chain)



Ground State



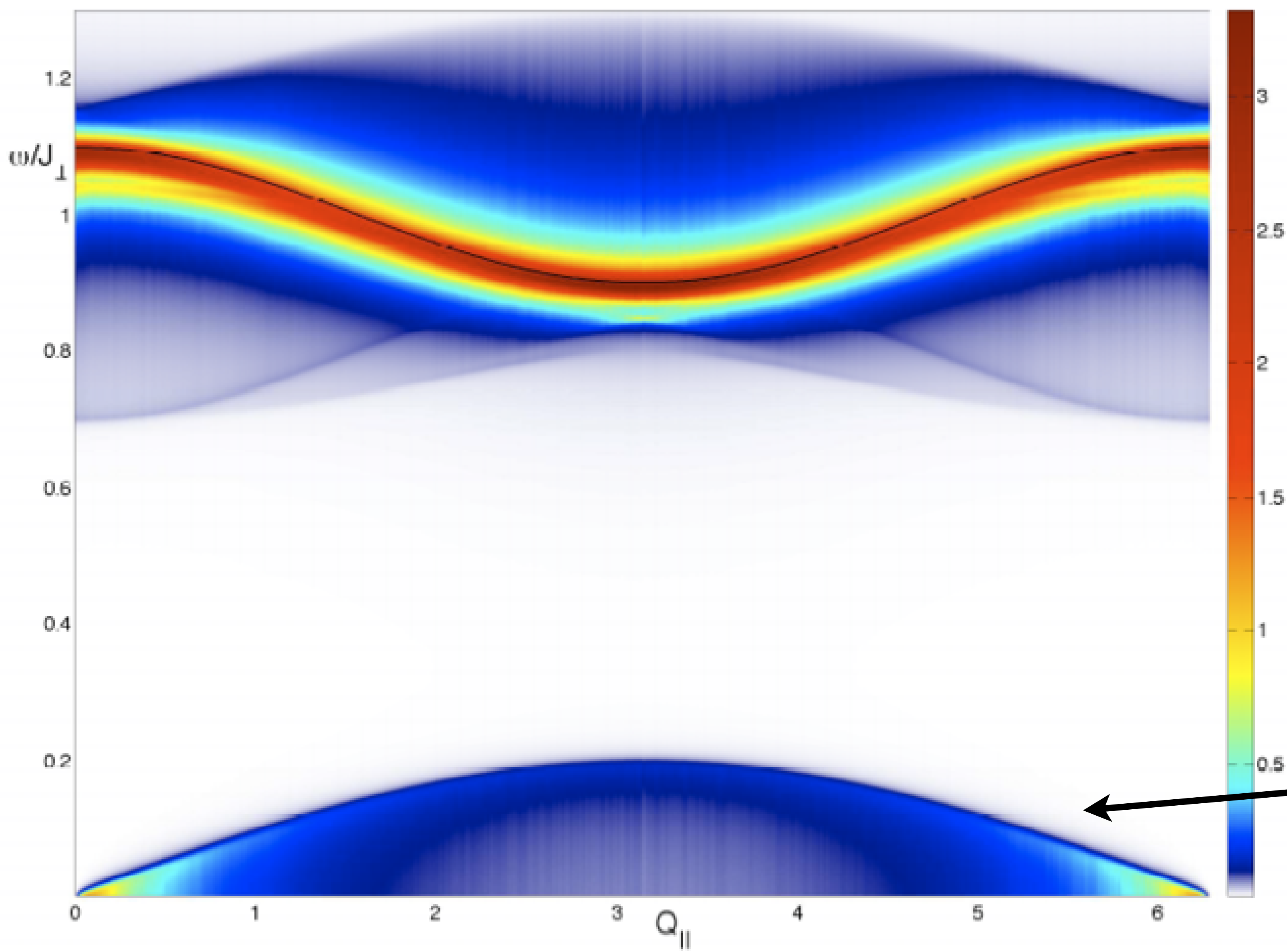
"Triplon" S=1 Excitation

Low-T expansion works.

For $J_{||} \ll J_\perp$ can still use $J_{||}=0$ quantum numbers (in PT)

→ combine low-T expansion with PT in $J_{||}$

Low-T DSF for 2-leg ladder: $J_{||}=0.1\ J_\perp$, $T=0.5\ J_\perp$, $Q_\perp=\pi/2$



Asymmetry in ω depends on Q

T>0 "Resonance"

Low-T Expansion Method

Input: **Energies E_n and matrix elements $\langle r|S^a(0,0)|s\rangle$**

Output: **Dynamical susceptibility at low T ($e^{-\beta\Delta} \ll 1$)**

Can apply it to:

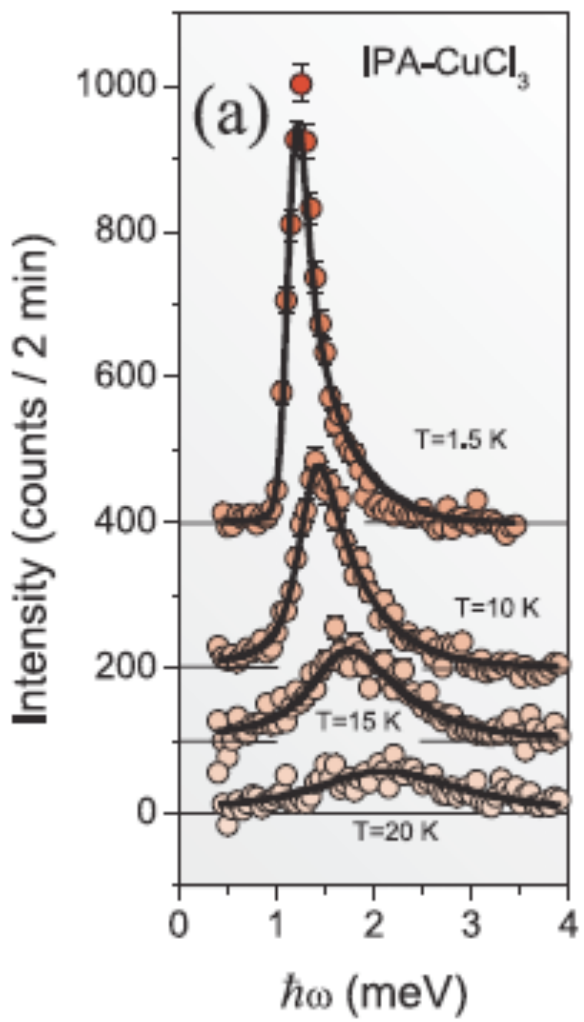
A. Integrable Models (know matrix elements exactly)
$\Rightarrow \Sigma(\omega,q,T)$ in terms of (simple) integrals

B. Models where we know the eigenstates to a good approximation

Linewidth and “T-dependent gap”

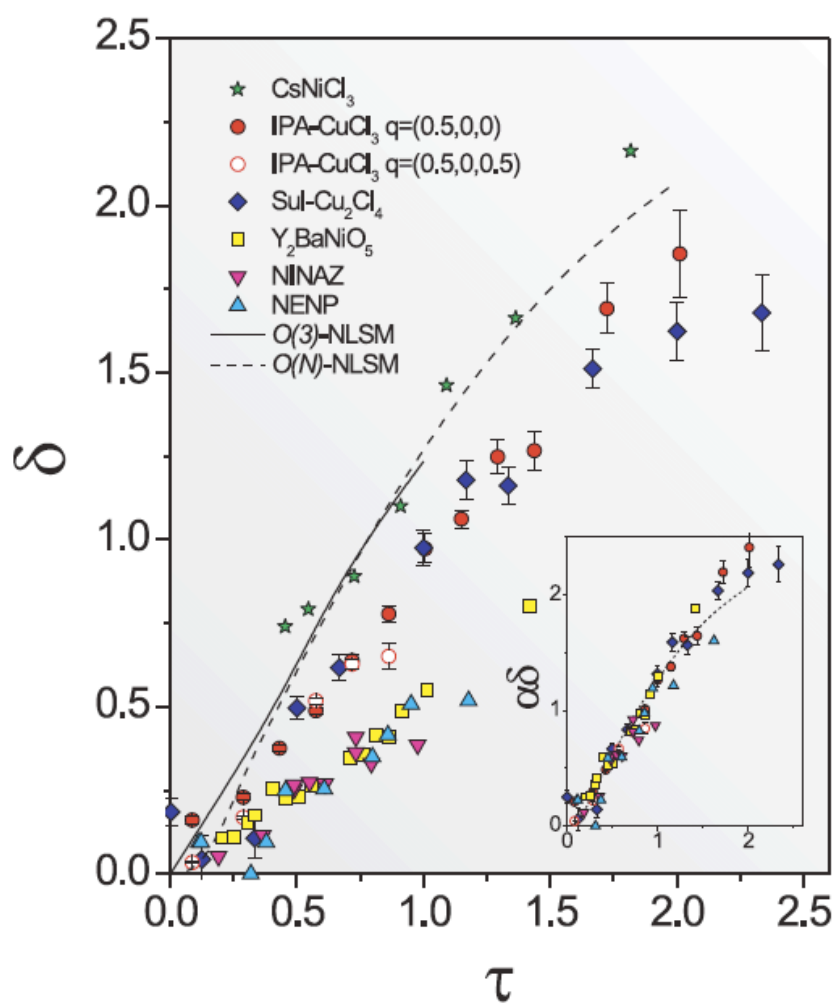
(A. Zheludev et al ’08)

T=0 Gap Δ« Magnon Bandwidth

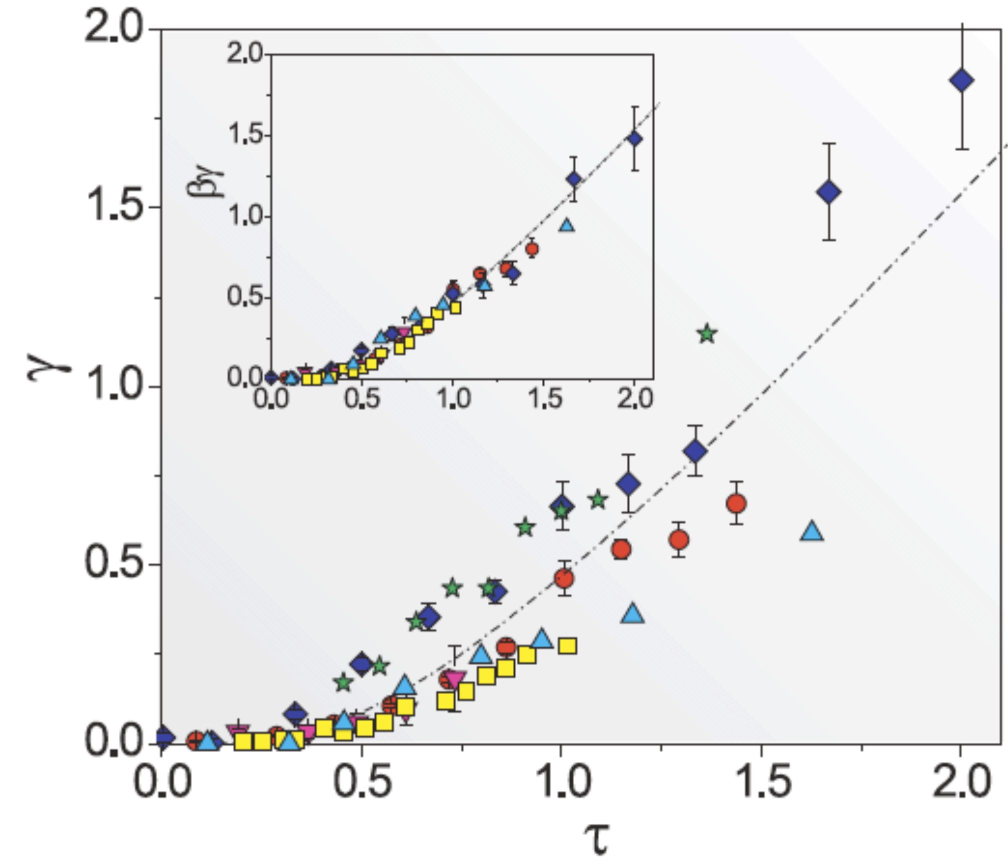


2-leg spin-1/2 ladder

“Gap” $\delta^2=(\Delta^2(T)/\Delta^2)-1$



Halfwidth (τ=T/Δ)



“quasi-universal” behaviour

Form Factor Bootstrap Approach

(**Karowski**/Weisz '78, **Smirnov** '93, **Lukyanov** '95, Delfino/Mussardo '95, Balog/Niedermaier '97, Babujian/Karowski '99...)

Exact S-matrix → **Matrix elements of local operators**

$\langle n|\, \mathcal{O}(0,0)\, |m\rangle$

Sigma Model is integrable → Can construct exact eigenstates labelled by n particle momenta

Basis States: $|n\rangle = |p_1,\ldots,p_n\rangle_{a_1\ldots a_n}$

Total Momentum: $P_n = \sum_{j=1} p_j$

Total Energy: $E_n = \sum_{j=1} \varepsilon(p_j) \geq n\Delta$

Form Factor Bootstrap Approach

(**Karowski**/Weisz '78, **Smirnov** '93, **Lukyanov** '95, Delfino/Mussardo '95, Babujian/Karowski '99...)

integrability

↓

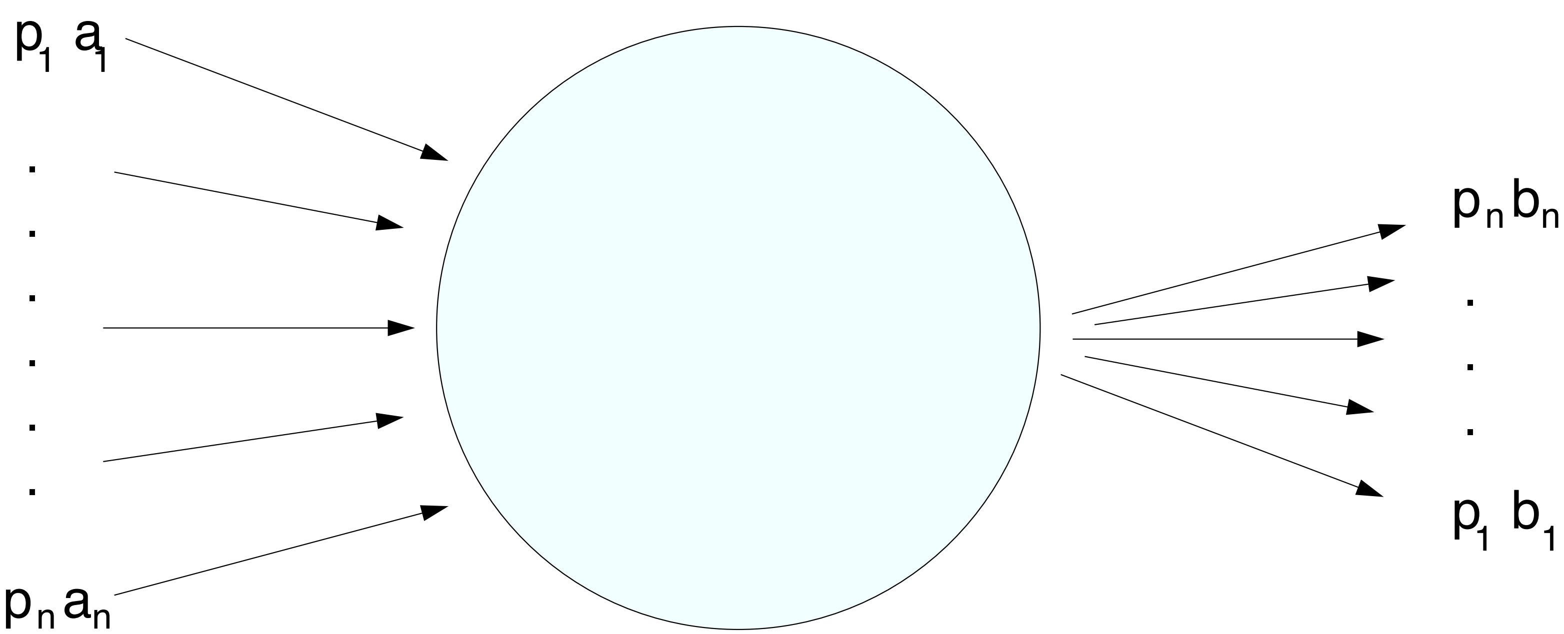
Matrix elements of local operators $\langle n| \mathcal{O}(0,0) |m\rangle$

Idea: Use Lehmann representation to calculate response functions

2-Point Functions in Integrable QFTs

e.g. O(3) nonlinear sigma model $\mathcal{L} = \frac{v}{2g}\partial_\mu \mathbf{n}\cdot\partial^\mu \mathbf{n}$ $\mathbf{n}\cdot\mathbf{n} = 1$

Purely elastic scattering ⇒ Hamiltonian eigenbasis of scattering states



Basis States: $|n\rangle = |p_{1,\dots,}p_n\rangle_{a_1\dots a_n}$

Total Momentum: $P_n = \Sigma_{j=1}\, p_j$

Total Energy: $E_n = \Sigma_{j=1}\, \varepsilon(p_j) \geq n\Delta$

Idea: Use Lehmann representation to calculate response functions

Field Theory Limit of Integer-S Heisenberg Chain

(Haldane '83
Affleck '89)

$$H = J \sum_n \mathbf{S}_n \cdot \mathbf{S}_{n+1} \ , \quad \mathbf{S}_n^2 = S(S+1)$$

large integer S, low energies

$$\mathbf{S}_j \approx S(-1)^{ja_0}\mathbf{n}(x) + \frac{1}{vg}\mathbf{n}(x) \times \frac{\partial \mathbf{n}(x)}{\partial t}$$

O(3) nonlinear sigma model

$$\mathcal{L} = \frac{v}{2g}\, \partial_\mu \mathbf{n} \cdot \partial^\mu \mathbf{n}$$

Constraint: n·n=1

$$v = JSa_0.$$
$$g = 2/S$$

Dynamical Structure Factor around q=π:

$$S(\omega, k = \frac{\pi}{a_0} + q) \propto \int dt\ dx\ e^{i\omega t - iqx} \langle 0| n^a(t,x) n^a(0,0)|0\rangle$$